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Fun with complex numbers

1. Express the following in "Cartesian form" $x + iy$, where x and y are real: $1/(2 - 3i)$, $(1 + 2i)/(3 + 4i)$, $5e^{6i}$. [2+2+2=6 pts.]

   Solution: For the first two use $1/z = z^*/z^*z$, and for the last use Euler's identity:
   $1/(2 - 3i) = (2 + 3i)/13 = (2/13) + i(3/13)$
   $(1 + 2i)/(3 + 4i) = (1 + 2i)(3 - 4i)/25 = (11 + 2i)/25 = (11/25) + i(2/25)$
   $5e^{6i} = 5\cos 6 + i5\sin 6$

2. Express the following in "polar form" $re^{i\varphi}$, where r is a real positive number and θ is real: -6, $-5i$, $(1 + i)/\sqrt{2}$, $2 - 3i$, $(2 + i)/(1 + 2i)$. (*Note*: Be careful to get the correct sign for the phase.) [2+2+2+2+2=10 pts.]

   Solution:
   $-6 = 6e^{i\pi}$
   $-5i = 5e^{-i\pi/2}$
   $(1 + i)/\sqrt{2} = e^{i\pi/4}$
   $2 - 3i = \sqrt{13}e^{i\tan^{-1}(-3/2)}$, where it is important that we choose the negative arctangent (≈ -0.9828) between 0 and $-\pi$, since $y < 0$.
   $(2 + i)/(1 + 2i) = (2 + i)(1 - 2i)/5 = (4 - 3i)/5 = e^{i\tan^{-1}(-3/4)} = e^{-i(0.6435)}$, where again we must take the arctangent between 0 and $-\pi$ since $y < 0$.
   Alternatively, note that $(2 + i)$ and $(1 + 2i)$ have the same modulus, so we can just take the ratio of their phases, $(2 + i)/(1 + 2i) = e^{i(\tan^{-1}(1/2)-\tan^{-1}(2))} = e^{-i(0.6435)}$.

3. (i) Find all the cube roots of -1, i.e. $(-1)^{1/3}$, and express them all in both polar form and in Cartesian form. (ii) Plot and label them in the complex plane. [10+5=15 pts.]

   Solution: (i) $(-1)^{1/3} = (e^{i\pi+i2\pi n})^{1/3} = e^{i\pi/3+i2\pi n/3} = -1, e^{i\pi/3}, e^{-i\pi/3}$. In Cartesian form these are -1, $1/2 + i\sqrt{3}/2$, $1/2 - i\sqrt{3}/2$. (ii) In the plane they lie on the unit circle, equally spaced by 120 degrees, with one of them at -1.

4. Show that there are infinitely many values of i^i and they are all real. (*Hint*: Remember the definition of the complex exponential: $w^z = \exp(z \ln w)$.) [5 pts.]

   Solution: $i^i = e^{i(i\pi/2+i2\pi n)} = e^{-\pi/2-2\pi n}$, where n is any integer.

5. Prove the trigonometric identities $\cos(a+b) = \cos a \cos b - \sin a \sin b$ and $\sin(a+b) = \sin a \cos b + \cos a \sin b$ by taking the real and imaginary parts of the identity $\exp(i(a+b)) = \exp(ia)\exp(ib)$. You may of course use the fact that $\exp(i\theta) = \cos\theta + i\sin\theta$. [5pts.]

Solution:
The lhs is $\exp(i(a+b)) = \cos(a+b) + i\sin(a+b)$. The rhs is

$$\begin{aligned} \exp(ia)\exp(ib) &= (\cos a + i \sin a)(\cos b + i \sin b) && (1) \\ &= (\cos a \cos b - \sin a \sin b) + i(\sin a \cos b + \cos a \sin b). && (2) \end{aligned}$$

Equating the real and imaginary parts of the lhs and rhs gives the trig. identities.

6. Express the real and imaginary parts of the following functions in terms of $x = Re(z)$ and $y = Im(z)$: z^3, e^z, e^{iz}, $\sin z$, $1/(z^2+1)$. [2+2+2+2+2=10 pts.]

Solution:
$z^3 = (x+iy)^3 = (x+iy)(x^2 - y^2 + 2ixy) = (x^3 - 3xy^2) + i(3x^2y - y^3)$.
$e^z = e^{x+iy} = e^x e^{iy} = (e^x \cos y) + i(e^x \sin y)$.
$e^{iz} = e^{i(x+iy)} = e^{ix}e^{-y} = (e^{-y}\cos x) + i(e^{-y}\sin x)$.
$\sin z = \sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy) = (\sin x \cosh y) + i(\cos x \sinh y)$.
$(z^2+1)^{-1} = (x^2 - y^2 + 1 + i2xy)^{-1} = \frac{x^2-y^2+1}{(x^2-y^2+1)^2+(2xy)^2} + i\frac{-2xy}{(x^2-y^2+1)^2+(2xy)^2}$.

7. Problem 16.1h [10 pts.]

Solution: Let $h(z) = z^2 = (x^2 - y^2) + i2xy$, so $f(z) = x^2 - y^2$ and $g(z) = 2xy$. Denote partial derivative with respect to x by the subscript "$,x$", and the same for y. Then $f_{,x} = 2x = g_{,y}$ and $f_{,y} = -2y = -g_{,x}$, which verifies the Cauchy-Riemann equations. Also $\nabla^2 f = f_{,xx} + f_{,yy} = 2 - 2 = 0$ and $\nabla^2 g = g_{,xx} + g_{,yy} = 0 + 0 = 0$. Finally $\nabla f \cdot \nabla g = f_{,x}g_{,x} + f_{,y}g_{,y} = (2x)(2y) + (-2y)(2x) = 0$. By the way note that also the magnitudes of the gradients of f and g are equal: $f_{,x}^2 + f_{,y}^2 = (2x)^2 + (-2y)^2$ while $g_{,x}^2 + g_{,y}^2 = (2y)^2 + (2x)^2$. The contours of f are hyperbolae with asymptotes $|y| = |x|$, while those of g are hyperbolae with asymptotes $x = 0$ and $y = 0$.