HW#2 —Phys374—Fall 2013 Prof. Alessandra Buonanno
Due before class, Thursday, Sep. 19, 2013 Room 4212, (301)405-1440
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1. In class we made a Taylor series expansion of $f(x) = 1/(1-x)$ about $x = 0$, and observed that this expansion seemed to converge only in the interval $|x| < 1$. Nevertheless, we may expand around points outside this interval. Find the Taylor series about the point $x = 3$. Write a formula for the nth term in the series. (It turns out that this series converges in the interval $|x - 3| < 2$.) [5 pts.]

   Solution: The first term in the series is $f(3) = (-2)^{-1}$. For the rest of the terms note that $f'(x) = (1-x)^{-2}$, $f''(x) = 2(1-x)^{-3}$, and $f^{(n)}(x) = n!(1-x)^{-n-1}$ for $n \geq 1$. Thus $f^{(n)}(3) = n!(-2)^{-n-1}$ for $n \geq 1$. The nth term is thus $(-2)^{-n-1}(x-3)^n$.

2. **Relativistic energy**

   The relativistic relation between energy E, 3-momentum p, and rest mass m is

$$E^2 = p^2c^2 + m^2c^4, \tag{1}$$

   where c is the speed of light. In class we found the first few terms in the expansion of the function $E(p)$ in powers of p/mc, $E = mc^2 + p^2/2m - p^4/8m^3c^2 + \ldots$. The first term is the rest energy, and the second has the form of nonrelativistic kinetic energy. Those terms would give a good approximation for low momentum, i.e. when $p \ll mc$.

   In this problem look at the opposite limit. Find the approximate form of $E(p)$ for very *high* momentum $p \gg mc$ (equivalently, $E \gg mc^2$). More precisely, expand in powers of mc/p, keeping only the m-independent term (which governs massless particles, like photons) and the leading order mass dependence. (Relativistic momentum is $mv/\sqrt{1 - v^2/c^2}$, so it can be much greater than mc when v is very close to c, and it can be nonzero in the limit $m \to 0$ only if $v \to c$.) [5 pts.]

   Solution:

$$\begin{aligned}
E &= (p^2c^2 + m^2c^4)^{1/2} && (2)\\
&= pc(1 + (mc/p)^2)^{1/2} && (3)\\
&= pc(1 + (1/2)(mc/p)^2 + O[(mc/p)^4]) && (4)\\
&\approx pc + m^2c^3/2p. && (5)
\end{aligned}$$

3. **Diatomic molecules**

The Lennard-Jones potential, a simple approximation for the interaction energy between two atoms separated by a distance r, takes the form

$$V(r) = V_0 \left[\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^{6} \right] \tag{6}$$

where V_0 and a are constants. The repulsive term arises from the Pauli exclusion principle, while the attractive term arises from the van der Waals interaction (induced charge dipole interactions).

(a) Find r_{min}, where the potential is minimum, and evaluate $V(r_{\text{min}})$. [3 pts.]

(b) Sketch the potential. Label V_0 and a on your graph. [2 pts.]

(c) Expand $V(r)$ around the minimum, keeping terms out to quadratic order in $r - r_{\text{min}}$. [5 pts.]

(d) The quadratic approximation to the potential near r_{min} is a harmonic oscillator potential, which governs vibrations of a pair of atoms bound in a molecule. If each atom has a mass m, what is the frequency of such harmonic vibrations, expressed as a function of m, V_0, and a? Verify that your result has the right dimensions. [3 pts.]

(e) For small vibrations the average atomic separation is r_{min}. As the amplitude of the vibrations gets larger, the fact that the potential is not simply quadratic becomes apparent. Does the time averaged separation become smaller, larger, or stay the same? Explain your answer! (*Note*: This is why crystals expand when heated.) [2 pts.]

Solution:

(a) To save writing, let's adopt units with $a = 1$ and $V_0 = 1$. We can put the a and V_0 dependence back in at the end using dimensional analysis. Thus $V(r) = r^{-12} - 2r^{-6}$. At the minimum we have $0 = V'(r) = -12r^{-13} + 12r^{-7}$, whose solution is $r = 1$, i.e. $r = a$. The value of the potential there is $V(1) = 1 - 2 = -1 = -V_0$.

(b) Sketch omitted for the moment.

(c) $V(r) \approx V(1) + V'(1)(r-1)^2 + (1/2)V''(1)(r-1)^2$. Now $V'(1) = 0$ and $V''(r) = (12)(13)r^{-14} - (12)(7)r^{-8}$, so $V''(1) = 72$. Hence $V(r) \approx -1 + (1/2)(72)(r-1)^2 = -V_0 + (1/2)(72V_0/a^2)(r-a)^2$.

(d) The effective spring constant $k = 72V_0/a^2$ is defined by comparing with the harmonic oscillator potential. The angular frequency of an oscillator is $\omega = (k/m)^{1/2} = (72V_0/ma^2)^{1/2}$. Since $V_0 \sim ML^2T^{-2}$, $m \sim M$, and $a \sim L$ this is $\sim (T^{-2})^{1/2} = T^{-1}$, as appropriate.

(e) The potential gets steeper than the harmonic oscillator for $r < a$ and shallower for $r > a$, so the system spends more time on the $r > a$ side when oscillating about the minimum, so the time averaged separation is *greater* than a.

4. **Speed of surface waves**

In HW#1 you used dimensional analysis to find the speed of surface waves on water. When surface tension is irrelevant, and gravity is the only restoring force, these are called "gravity waves". The result was $v \propto \sqrt{gh}$ for shallow water ($h \ll \lambda$) and $v \propto \sqrt{g\lambda}$ for deep water ($\lambda \ll h$). It turns out that the formula interpolating between these two limits is

$$v_{ph} = \sqrt{\frac{g \tanh(kh)}{k}}, \tag{7}$$

where $k = 2\pi/\lambda$ is the wave vector. Here $\tanh(x)$ is the hyperbolic tangent defined as

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh(x)}{\cosh(x)}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}, \tag{8}$$

where $\sinh(x)$ and $\cosh(x)$ are the hyperbolic sine and cosine.

(a) Obtain the Taylor expansion of $\tanh(x)$ for small $x \ll 1$ up through the order of x^3. You can use expansions (3.14) and (3.15) from the textbook, or take derivatives of $\tanh(x)$ straightforwardly for Eq. (3.11). [3 pts.]

(b) For $x \gg 1$, show that $\tanh(x) \approx 1 - 2e^{-2x} + O(e^{-4x})$. [3 pts.]

(c) Using the approximations for $\tanh(x)$ derived in parts 4a and 4b, show that Eq. (7) reduces to the forms of v quoted above in the shallow $kh \ll 1$ and deep $kh \gg 1$ limits in the leading order. [4 pts.]

Solution:
(a)

$$\begin{aligned} \tanh x &= \sinh x / \cosh x && (9) \\ &= [x + x^3/6 + O(x^5)]/[1 + x^2/2 + O(x^4)] && (10) \\ &= [x + x^3/6 + O(x^5)][1 - x^2/2 + O(x^4)] && (11) \\ &= x - x^3/3 + O(x^5). && (12) \end{aligned}$$

(b)

$$\begin{aligned} \tanh x &= (e^x - e^{-x})/(e^x + e^{-x}) && (13) \\ &= (1 - e^{-2x})/(1 + e^{-2x}) && (14) \\ &= (1 - e^{-2x})[1 - e^{-2x} + O(e^{-4x}] && (15) \\ &= 1 - 2e^{-2x} + O(e^{-4x}). && (16) \end{aligned}$$

(c) If $kh \ll 1$ then $\tanh kh \approx kh$, so $v_{ph} \approx \sqrt{gkh/k} = \sqrt{gh}$.
If $kh \gg 1$ then $\tanh kh \approx 1$, so $v_{ph} \approx \sqrt{g/k} = \sqrt{g\lambda/2\pi}$.

5. Find an approximation to the root of $ay^3 + y + 2 = 0$ near $y = -2$, as an expansion in a up through $O(a^2)$, assuming $a \ll 1$. (Don't solve it exactly, even if you can. *Tip*: Since y^3 is multiplied by a, you need only keep the terms up to $O(a)$ in y^3.) [10 pts.]

Solution: When $a = 0$ the root is $y = -2$. For any a, suppose the root has the form of an expansion $y = -2 + y_1 a + y_2 a^2 + O(a^3)$. Then $y^3 = -8 + 12 y_1 a + O(a^2)$, so

$$ay^3 + y + 2 = (y_1 - 8)a + (12y_1 + y_2)a^2 + O(a^3). \tag{17}$$

Setting this to zero order by order in a yields $y_1 = 8$ and $y_2 = -96$.

6. **Lagrange points L1 and L2**

If the earth's orbit around the sun were perfectly circular, there would be a point L1 inside the earth's orbit where a satellite (of negligible mass) would orbit the sun with precisely the same angular velocity as the earth, since the extra pull of the sun on the closer satellite would be canceled by the pull of the earth (the gravitational pull of all other bodies in the solar system being neglected). There would be a similar point L2 just outside the earth's orbit where the smaller pull of the sun would be supplemented by the pull of the earth. These are called "Lagrange points".

Setting the force on the earth equal to the mass times the centripetal acceleration of the earth we have $Gm_s m_e / r_s^2 = m_e r_s \omega_e^2$, where m_s and m_e are the masses of the sun and the earth, r_s is the distance from sun to earth, and ω_e is the angular velocity of the earth in its orbit. Hence

$$\omega_e^2 = \frac{Gm_s}{r_s^3}. \tag{18}$$

For the satellite at L1 we have

$$\frac{Gm_s m_{sat}}{(r_s - d_1)^2} - \frac{Gm_e m_{sat}}{d_1^2} = m_{sat}(r_s - d_1)\omega_{sat}^2, \tag{19}$$

where d_1 is the distance from the earth to L1. Hence

$$\omega_{sat}^2 = \frac{Gm_s}{(r_s - d_1)^3} - \frac{Gm_e}{d_1^2(r_s - d_1)}. \tag{20}$$

Setting $\omega_e = \omega_{sat}$ then yields

$$\frac{m_s}{r_s^3} = \frac{m_s}{(r_s - d_1)^3} - \frac{m_e}{d_1^2(r_s - d_1)}. \tag{21}$$

The result for L2 is obtained just by the replacements $m_e \to -m_e$ and $d_1 \to -d_2$ in this expression.

(a) Re-write the relation (21) using only the dimensionless ratios $\mu = m_e/m_s$ and $\delta = d_1/r_s$. [1 pt.]

(b) (i) Find the solution for δ to leading order in μ. (ii) Show that the result for L2, $\delta_2 = d_2/r_s$, is identical. (*Suggestion*: First express the equation as a fifth order polynomial equation, and then ascertain which terms give the leading order solution. *Answer*: $\delta \approx (\mu/3)^{1/3}$. *Trick*: Note that the equation for L2 is obtained from that for L1 by the substitutions $\delta \to -\delta_2$ and $\mu \to -\mu$. Making these substitutions in your solution for δ you can easily find δ_2.) [5 pts.]

Using the mass ratio $\mu = 1/332,830$, one finds $\delta \approx 10^{-2}$, to a very good approximation, by chance. The distance to L1 and L2 is thus about $1/100$ times the distance to the sun. The (mean) solar distance is $d_s \simeq 1.5 \times 10^8$ km, so $d_{1,2} \simeq 1.5 \times 10^6$ km. This is about 4 times the distance to the moon, and about 235 times the radius of the earth.

(c) Show that the *next to leading* order term in $\delta(\mu)$ for L1 is $-3^{-5/3}\mu^{2/3}$, and for L2 it is $+3^{-5/3}\mu^{2/3}$. How large is this correction compared to the leading order term? [4 pts.]

Solution:

(a)

$$1 = \frac{1}{(1-\delta_1)^3} - \frac{\mu}{\delta_1^2(1-\delta_1)}. \tag{22}$$

(b) (i) Multiplying (22) by $\delta^2(1-\delta)^3$ yields

$$\delta^2(1-\delta)^3 = \delta^2 - \mu(1-\delta)^2, \tag{23}$$

or

$$-3\delta^3 + 3\delta^4 - \delta^5 = -\mu(1 - 2\delta + \delta^2). \tag{24}$$

When $\mu = 0$, i.e. neglecting the mass of the earth, the only solution is $\delta = 0$. Hence we might try a series solution $\delta = \delta_1\mu + \delta_2\mu^2 + \ldots$. However, this will not work, since no terms involving just one power of μ will appear to cancel the $-\mu$ on the right hand side. Evidently this is a case of "singular perturbation theory". At lowest order the equation reduces to $-3\delta^3 = -\mu$, hence $\delta \approx (\mu/3)^{1/3}$. That is, the leading order μ dependence goes like the cube root of μ. (ii) Making the substitutions indicated in the *Trick* we get $-\delta_2 \approx (-\mu/3)^{1/3}$. Since $(-1)^{1/3} = -1$, this is the same as $\delta_2 \approx (\mu/3)^{1/3}$.

(c) To find the next to leading order term, we assume a series in powers of $\mu^{1/3}$: $\delta = 3^{-1/3}\mu^{1/3} + \delta_2\mu^{2/3} + O(\mu)$. Inserting this into (24) and keeping explicitly only the terms through $O(\mu^{4/3})$ we have

$$-3(3^{-1/3}\mu^{1/3} + \delta_2\mu^{2/3})^3 + 3(3^{-4/3}\mu^{4/3}) = -\mu + 2(3^{-1/3}\mu^{4/3}) + O(\mu^{5/3}) \tag{25}$$

The $O(\mu)$ part is satisfied automatically by our choice of the lowest order part of δ. Equating the coefficients of the $O(\mu^{4/3})$ on the left and right hand sides yields

$$-3^2 3^{-2/3}\delta_2 + 3^{-1/3} = 2(3^{-1/3}), \tag{26}$$

whose solution is $\delta_2 = -3^{-5/3}$. Thus

$$\delta = 3^{-1/3}\mu^{1/3} - 3^{-5/3}\mu^{2/3} + O(\mu). \tag{27}$$

Using the substitution trick we then have $-\delta_2 = 3^{-1/3}(-\mu)^{1/3} - 3^{-5/3}(-\mu)^{2/3} + O(\mu)$, or $\delta_2 = 3^{-1/3}(\mu)^{1/3} + 3^{-5/3}(\mu)^{2/3} + O(\mu)$. The ratio of consecutive terms is of order $\mu^{1/3} \sim 10^{-2}$.