HW#10 —Phys374—Fall 2013 Prof. Alessandra Buonanno
Due before class, Tuesday, Dec. 3, 2013 Room 4212, (301)405-1440
http://www.physics.umd.edu/rgroups/grt/buonanno/Phys374/ buonanno@umd.edu

1. Consider the "rectified cosine function" defined by

$$f(x) = \cos(\pi x/2L), \qquad L \leq x \leq L, \tag{1}$$

and continued periodically so that $f(x + 2L) = f(x)$. [2+3+5+5=15 pts.]

(a) Sketch the function $f(x)$ over several periods.

(b) Use the symmetry to explain why the Fourier coefficients b_n vanish.

(c) Find the non-vanishing Fourier coefficients. (*Hints*: (i) To clean things up, change variables to $\theta = \pi x/L$. (ii) You'll need to do a probably unfamiliar integral, which you can look up or work out for yourself.)

(d) Using a computer program (Mathematica, Maple, Matlab, or something else) plot the sum of the first few terms in the Fourier series, together with (1), for $\theta \in (-2\pi, 2\pi)$. Show the result with 1 (just the constant part), 2, 5, and 20 terms included. With 5 terms the sum should already be quite close to (1), except near the zeros where the slope is discontinuous.

Solution: (a) See fourier.pdf link. (b) The function is even under $x \to -x$, so the integral of its product with an odd function such as $\sin m\pi x/L$ vanishes. (c)

$$a_n = L^{-1} \int_{-L}^{L} \cos(\pi x/2L) \cos(n\pi x/L)\, dx \tag{2}$$

$$= \pi^{-1} \int_{-\pi}^{\pi} \cos(\theta/2) \cos(n\theta)\, d\theta \tag{3}$$

$$= \pi^{-1}(-1)^n/(\frac{1}{4} - n^2). \tag{4}$$

Thus $a_0 = 4/\pi$, $a_1 = 4/3\pi$, $a_2 = -4/15\pi$, etc. (d) See fourier.pdf link. (The functions are plotted there from -3π to 3π, and 50 terms are plotted there, rather than 20.)

2. Find the Fourier transform of $f(t) = A\sin(\omega_0 t + \varphi)$. [10 pts.]

Solution:

$$\tilde{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt \tag{5}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A}{2i} \Big(e^{i(\omega_0 t+\varphi)} - e^{-i(\omega_0 t+\varphi)} \Big)\, e^{i\omega t}\, dt \tag{6}$$

$$= \frac{-iA}{2} \Big(e^{i\varphi}\delta(\omega+\omega_0) - e^{-i\varphi}\delta(\omega-\omega_0) \Big). \tag{7}$$

(8)

3. Problems 15.6 g,h (Fourier transform of correlation and Parseval's theorem) [10 pts.] (*Note*: The conventions (15.42), (15.43) are used here.)

Solution: Leaving off the $\pm\infty$ limits of integration,

$$\int d\omega\, e^{-i\omega t} F(\omega) H^*(\omega) = \frac{1}{(2\pi)^2} \int d\omega\, e^{-i\omega t} \int dt'\, e^{i\omega t'}\, f(t') \int dt''\, e^{-i\omega t''}\, h^*(t'') \quad (9)$$

$$= \frac{1}{2\pi} \int dt' dt''\, \delta(-t + t' - t'')\, f(t')\, h^*(t'') \quad (10)$$

$$= \frac{1}{2\pi} \int dt''\, f(t + t'')\, h^*(t'') \quad (11)$$

Part (h) then follows just as the book says, putting $t = 0$ and setting the two functions equal.