Note: Be sure to fully justify the relation between the given integral and the closed contour integral you employ in the following problems.

1. Find the residues of the following functions at the given values of $z$ [3 × 5 = 15 pts.]:

   (a) $(z + z^2)^{-1}$ at 0 and at $-1$.
   (b) $z^{-2} \ln(1 + 2z)$ at 0
   (c) $[z^3(z + 2)^2]^{-1}$ at 0 and at $-2$.
   (d) $\cos z/(2z - \pi)^4$ at $\pi/2$
   (e) $(z^2 + 1)^{-3}$ at $\pm i$.

2. (a) Show using contour integration that

   $$\int_0^\infty \frac{\cos mx}{x^2 + a^2} \, dx = \frac{\pi}{2a} e^{-ma}$$

   Hint: See the similar example in the textbook.

   (b) Explain in words why the result decays so rapidly (i) as $m$ grows with $a$ fixed, and (ii) as $a$ grows with $m$ fixed. [8+2=10 pts.]

3. (a) Evaluate the integral $\int_0^\infty dx/(x^n + 1)$, where $n \geq 2$ is a positive integer, by relating it to the contour integral around the boundary of an infinite piece of pie with edges $\theta = 0$ and $\theta = 2\pi/n$, together with the arc at infinity that joins these edges. (b) Show that the result approaches 1 as $n \to \infty$, and explain with reference to the behavior of the integrand why this is the limiting value. (Answer: $$(\pi/n)/\sin(\pi/n).)$$ [8+2=10 pts.]

4. Consider potential flow perpendicular to an infinite solid cylinder of radius $R$. This reduces to a two-dimensional problem in the $xy$ plane. The cylinder intersects the plane in a disk. For the boundary condition “at infinity”, suppose that far from the cylinder in all directions the velocity is $v = v_0 \hat{x}$. The boundary condition at the cylinder surface is that there is no flow perpendicular to the cylinder, so $\mathbf{v} \cdot \hat{r} = 0$, taking the origin of polar coordinates at the center of the cylinder. That is, the partial derivative of the velocity potential with respect to radius $r$ (at fixed angle) vanishes at $r = R$. To solve for the flow we need only find an analytic function of $z = x + iy$ whose real part satisfies the appropriate boundary conditions.

   (a) Find an analytic function $h_1(z)$ whose real part is a potential for the velocity field $v = v_0 \hat{x}$.

   (b) Find an analytic function $h_2(z)$, to be added to your function $h_1(z)$ from the previous part, such that the real part of $h(z) = h_1(z) + h_2(z)$ satisfies the boundary condition everywhere on the cylinder, as well as the boundary condition at infinity. To do this assume $h_2(z) = az^n$, and find the values for constants $a$ and
n for which the boundary conditions are satisfied everywhere on the cylinder. 
(*Hint:* Use polar coordinates. The result, which you are supposed to derive, is
\[ h(z) = v_0(z + R^2/z). \]

(c) Using results from Homework 7, find the velocity (magnitude and direction) at
the point \( (x, y) = (0, R) \) on the surface of the cylinder. How does it compare
with \( v_0 \)?

(d) Find the equation for the flow line that goes through the point \( (x, y) = (0, y_0) \)
(with \( y_0 > R \)). (The equation should involve \( x, y, y_0, v_0 \) and \( R \)).

(e) Find the asymptotic \( y \) value \( y_\infty \) when \( x \to \infty \) for the flow line of problem 4d.

(f) Sketch the flow lines of problem 4d for the cases \( y_0 = R \) and \( y_0 = 2R \), along with
the circle of radius \( R \) representing the cross-section of the cylinder.