1. (a) Evaluate $\int z\,dz$ along the following two contours connecting -1 to 1: (i) from -1 to 1 along the real axis, and (ii) along the semicircle of radius 1 in the complex plane. (Do both contour integrals explicitly. Use the polar angle as the variable of integration for (ii), and use $x = Re(z)$ as the variable of integration for (i).) Explain how you could have known the two integrals would agree without even evaluating them. (b) Repeat part (a) for the integral $\int z^*\,dz$, and explain why the two contour integrals do *not* agree in this case. [7+3=10 pts.]

2. **Fluid flow and analytic functions**: Problem 16.3 c,d,e,f [2+3+3+2=10 pts.]

3. **Vortex**: The velocity potential for a point source of fluid flow is given by the real part of $h(z) = k \ln z$ (where k is a constant), as shown in the previous problem. Show that the *imaginary part* of $h(z)$ is the velocity potential for a point vortex. Do this by showing that the flow lines are circles centered on $z = 0$, and find the relation between k and the circulation. [5 pts.]

4. (a) Show that a flow with complex velocity potential $h(z)$ has speed $|dh/dz|$. (*Hint*: Use the fact that $dh/dz = \partial_x h$, and use the Cauchy-Riemann equations. (b) Apply this to the previous two problems to find the flow speed as a function of r.
[8+2=10 pts.]

5. Consider potential flow in the wedge $0 < \theta < \alpha$ of the plane, bounded by walls at $\theta = 0$ (the x axis) and $\theta = \alpha$. At the walls the velocity must be parallel to the walls. (a) Show that the velocity potential $h(z) = Az^{\pi/\alpha}$ satisfies this boundary condition. (b) Sketch the flow lines for this potential for the cases $\alpha = \pi/2$, $\alpha = \pi$, $\alpha = 3\pi/2$, and $\alpha = 2\pi$. (c) Find the speed of the flow as a function of position on the plane for general α. Explain why your result makes sense for the case $\alpha = \pi$. (d) If $\alpha < \pi$, the speed goes to infinity as r goes to infinity. Explain how this is compatible with incompressibility of the flow. [3+3+2+2=10 pts.]

   Note that if $\alpha < \pi$ the speed goes to zero at the vertex of the wedge, whereas if $\alpha > \pi$ the speed goes to infinity at the vertex. I read that this is why the wind whistles when going over a pointed obstacle: locally supersonic velocities are reached.

   (*one more problem ...*)

6. Consider the real integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$$

where a and b are positive real numbers.

(a) Evaluate the integral using contour integration assuming $a \neq b$ (so there are only simple poles).

(b) Evaluate the integral using contour integration assuming $a = b$ from the beginning (so the poles are of order 2), and then check that you recover the same result by setting $a = b$ in the result of the previous part.

Hint: You can check your result by testing for a few properties: the integral is manifestly positive and symmetric under interchange of a and b, and scales as λ^{-3} under the scaling $a \to \lambda a$ and $b \to \lambda b$. (Equivalently, if you think of a and b as having dimensions, the integral has the same dimensions as a^{-3}.) [5+5=10 pts.]