1. (a) Evaluate \( \int z \, dz \) along the following two contours connecting \(-1\) to \(1\): (i) from \(-1\) to \(1\) along the real axis, and (ii) along the semicircle of radius 1 in the complex plane. (Do both contour integrals explicitly. Use the polar angle as the variable of integration for (ii), and use \( x = \text{Re}(z) \) as the variable of integration for (i).) Explain how you could have known the two integrals would agree without even evaluating them. (b) Repeat part (a) for the integral \( \int z^* \, dz \), and explain why the two contour integrals do not agree in this case. \([7+3=10\ pts.]\)

2. Fluid flow and analytic functions: Problem 16.3 c,d,e,f \([2+3+3+2=10\ pts.]\)

3. Vortex: The velocity potential for a point source of fluid flow is given by the real part of \( h(z) = k \ln z \) (where \( k \) is a constant), as shown in the previous problem. Show that the imaginary part of \( h(z) \) is the velocity potential for a point vortex. Do this by showing that the flow lines are circles centered on \( z = 0 \), and find the relation between \( k \) and the circulation. \([5\ pts.]\)

4. (a) Show that a flow with complex velocity potential \( h(z) \) has speed \( |dh/dz| \). \((\text{Hint:}\) Use the fact that \( dh/dz = \partial_x h_i \), and use the Cauchy-Riemann equations. (b) Apply this to the previous two problems to find the flow speed as a function of \( r \). \([8+2=10\ pts.]\)

5. Consider potential flow in the wedge \( 0 < \theta < \alpha \) of the plane, bounded by walls at \( \theta = 0 \) (the \( x \) axis) and \( \theta = \alpha \). At the walls the velocity must be parallel to the walls. (a) Show that the velocity potential \( h(z) = Az^{\pi/\alpha} \) satisfies this boundary condition. (b) Sketch the flow lines for this potential for the cases \( \alpha = \pi/2, \alpha = \pi, \alpha = 3\pi/2, \) and \( \alpha = 2\pi \). (c) Find the speed of the flow as a function of position on the plane for general \( \alpha \). Explain why your result makes sense for the case \( \alpha = \pi \). (d) If \( \alpha < \pi \), the speed goes to infinity as \( r \) goes to infinity. Explain how this is compatible with incompressibility of the flow. \([3+3+2+2=10\ pts.]\)

Note that if \( \alpha < \pi \) the speed goes to zero at the vertex of the wedge, whereas if \( \alpha > \pi \) the speed goes to infinity at the vertex. I read that this is why the wind whistles when going over a pointed obstacle: locally supersonic velocities are reached.

(\text{one more problem ...})
6. Consider the real integral
\[ \int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)} \]
where \(a\) and \(b\) are positive real numbers.

(a) Evaluate the integral using contour integration assuming \(a \neq b\) (so there are only simple poles).

(b) Evaluate the integral using contour integration assuming \(a = b\) from the beginning (so the poles are of order 2), and then check that you recover the same result by setting \(a = b\) in the result of the previous part.

*Hint:* You can check your result by testing for a few properties: the integral is manifestly positive and symmetric under interchange of \(a\) and \(b\), and scales as \(\lambda^{-3}\) under the scaling \(a \rightarrow \lambda a\) and \(b \rightarrow \lambda b\). (Equivalently, if you think of \(a\) and \(b\) as having dimensions, the integral has the same dimensions as \(a^{-3}\).) [5+5=10 pts.]