HW#3 —Phys374—Fall 2013 Prof. Alessandra Buonanno
Due before class, Thursday, Sep. 26, 2013 Room 4212, (301)405-1440
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1. **Equation of motion for a stretched string**

A stretched string will vibrate when plucked. If the string is finite it oscillates, while if the string is infinite it supports traveling waves. The physical parameters defining the problem are the mass per unit length ρ and tension τ of the string.

a) Show that the combination $\sqrt{\tau/\rho}$ has dimensions of velocity, *and* that one cannot make a dimensionless quantity using ρ, τ, and the wavelength λ. This allows us to infer that the wave speed is independent of wavelength and is proportional to $\sqrt{\tau/\rho}$. [4 pts.]

b) If the string has fixed endpoints separated by a length ℓ then it can vibrate at a particular set of normal mode frequencies. The lowest frequency must be proportional to some combination of the available constants ρ, τ, and ℓ. Find this combination. [4 pts.]

We derived in class the equation for a streched string:

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0, \tag{1}$$

where $v = \sqrt{\tau/\rho}$. This is a second order, linear partial differential equation. It is a wave equation for waves that travel at speed v. We showed in class that if the above equation has to be satisfied, the following equations must be satisfied:

$$\left(\frac{\partial}{\partial t} \pm v \frac{\partial}{\partial x}\right) y = 0 \tag{2}$$

with either the plus sign or the minus sign. Solutions are given by

$$y(x,t) = f(x - vt) + g(x + vt) \tag{3}$$

where f and g are arbitrary functions of their arguments.

c) The above reasoning implies that (3) satisfies the wave equation (1). Verify this explicitly by direct substitution of (3) into (1). [3 pts.]

If the endpoints are fixed and separated by a length ℓ, the string motion is still governed by the wave equation, but at the endpoints we have the restriction $y(0,t) = y(\ell,t) = 0$. These boundary conditions can be satisfied by appropriate superpositions of right and left moving traveling waves that cancel out to zero at the two endpoints.

d) Among the solutions with fixed endpoints are the *normal modes.* These are solutions in which the time dependence is given by an overall sinusoidal factor with frequency ω:

$$y(x,t) = \sin(\omega t + \varphi) f(x) \tag{4}$$

(i) Insert this form into the wave equation (1) and determine the ordinary differential equation satisfied by $f(x)$. [3 pts.] (ii) Find the general solution to this equation for

$f(x)$. [3 pts.] (iii) Impose the boundary conditions $y(0,t) = y(\ell,t) = 0$ to determine a discrete set of allowed frequencies ω, and the corresponding profiles $f(x)$. Any motion of the string is described by a sum of such normal mode solutions. [4 pts.]

2. **Convergence of improper integrals**

   (a) Show that $\int_1^\infty dt\, t^n$ is finite if and only if $n < -1$.

   (b) Show that $\int_0^\infty dt(a+bt)^n$, with $a, b > 0$, is finite if and only if $n < -1$.

   Be careful to treat the $n = -1$ cases properly. [5 pts.]

3. **Particle subjected to a drag force**

   Consider a particle of mass m in one dimension with a positive velocity v, acted on by a force that depends on the velocity as $-bv^n$, where b is a positive constant and n is a positive dimensionless number. This force acts to slow the particle down.

   (a) Use dimensional analysis to find an expression for how (i) the time for the particle to come to rest, and (ii) the distance it travels before coming to rest, can depend on the initial velocity v_0, together with m, b, and n. [5 pts.]

   (b) By integrating Newton's law, determine for which values of n the particle comes to rest in a finite time, and determine that time. Compare with part 3a. [5 pts.]

   (c) Determine for which values of n the particle travels a finite total distance before coming to rest (whether or not it actually stops in a finite time). Find an expression for that distance and compare with your result from part 3a. [5 pts.]