

Lorentz-Invariant Quantum Fields in the Spacetime Tangent Bundle

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A maximal-acceleration invariant quantum field is defined on the spacetime tangent bundle, with vanishing eigenvalue when acted on by the Laplace-Beltrami operator of the bundle, and the case is addressed in which the spacetime is Minkowskian, and the field is Lorentz invariant. In this case, the field is shown to be automatically regularized at the Planck scale, and particle spectra are cut off at extremely high energies. Microcausality is addressed by calculating the appropriate field commutators, and it is shown that, provided the adjoint field is consistently generalized, the necessary commutators are vanishing, and the field is microcausal, but that there are Planck-scale modifications of the boundary of the causal domain that are significant for extremely large relative four-velocities between the separated spacetime points. For vanishing relative four-velocity, the causal domain is canonical. The geometry of the causal domain indicates that near the Planck scale, causal connectivity may occur between spacelike separated points, and also at larger scales for extremely large relative four-velocities.

I. INTRODUCTION

Consider a maximal-acceleration invariant scalar field ϕ , defined on the spacetime tangent bundle, and having a vanishing eigenvalue when acted on by the invariant Laplace-Beltrami operator of the bundle, namely [1–5],

$$G^{-1/2} \frac{\partial}{\partial x^M} \left(G^{1/2} G^{MN} \frac{\partial}{\partial x^N} \phi \right) = 0. \quad (1)$$

Here, G_{MN} is the bundle metric tensor in a coordinate basis, and a point in the bundle manifold has coordinates

$$\begin{aligned} \{x^M; M = 0, 1, \dots, 7\} &= \{x^\mu, x^m; \mu = 0, 1, 2, 3; m = 4, 5, 6, 7\} \\ &\equiv \{x^\mu, \rho_0 v^\mu; \mu = 0, 1, 2, 3\}, \end{aligned} \quad (2)$$

where x^μ and v^μ are the spacetime and four-velocity coordinates, respectively. (Greek indices refer to spacetime, the base manifold of the bundle, and range from 0 to 3; lower case Latin indices refer to four-velocity space, the fiber manifold of the bundle, and range from 4 to 7; and upper case Latin indices refer to a point in the bundle and range from 0 to 7.) The length ρ_0 is of the order of the Planck length and is given by [6,7]

$$\rho_0 = \frac{c^2}{a_0} = \frac{(\hbar G/c^3)^{1/2}}{2\pi\alpha}, \quad (3)$$

where a_0 is the maximum possible proper acceleration relative to the vacuum [8], c is the velocity of light in vacuum, \hbar is Planck's constant divided by 2π , G is the universal gravitational constant, and α is a dimensionless constant of order unity.

The bundle structure, upon which the Laplace-Beltrami operator in Eq. (1) is based, follows from the invariance of maximal proper acceleration [6–8] which restricts the differential geometric structure of the tangent bundle of spacetime. For simplicity, the present work addresses a scalar field defined on the bundle and for which the invariant Laplace-Beltrami operator of the bundle acting on the field has a vanishing eigenvalue.

Equation (1) is invariant under the maximal acceleration group, the group of bundle-coordinate diffeomorphisms in eight dimensions [7, 9–11], and may determine possible particle excitations in the spacetime tangent bundle. The ordinary inhomogeneous Lorentz group (Poincaré group) is of course a subgroup of the maximal acceleration group, when the spacetime is Minkowskian. A connection with canonical relativistic quantum field theory [12] can be made by considering the case in which spacetime is flat and the spacetime metric is Minkowskian:

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (4)$$

The bundle metric then becomes

$$G^{MN} = \begin{bmatrix} \eta_{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{bmatrix}, \quad (5)$$

with the Minkowski metric appearing in both the base manifold and the fiber. If one substitutes Eq. (5) in Eq. (1), the scalar field equation reduces to the following eight-dimensional wave equation:

$$(\square_x + \rho_0^{-2} \square_v) \phi = 0, \quad (6)$$

where \square_x is the flat-spacetime d'Alembertian operator,

$$\square_x = \eta^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu}, \quad (7)$$

and \square_v is the analogous d'Alembertian operator defined on four-velocity space,

$$\square_v = \eta^{\mu\nu} \frac{\partial^2}{\partial v^\mu \partial v^\nu}. \quad (8)$$

II. FIELD MODES

Possible separable single-mode solutions $\phi_p(x, v)$ to Eq. (6), have the form

$$\phi_p(x, v) = \phi_1(x) \phi_2(v), \quad (9)$$

where the dependence on the spacetime coordinates x is separated from the dependence on the four-velocity coordinates v . If one substitutes Eq. (9) in Eq. (6), then for nonvanishing $\phi_1(x) \phi_2(v)$, one obtains

$$\frac{\square_x \phi_1(x)}{\phi_1(x)} + \rho_0^{-2} \frac{\square_v \phi_2(v)}{\phi_2(v)} = 0. \quad (10)$$

Note that the first term of Eq. (10) depends only on x , and the second term depends only on v , and therefore both terms must be given by constants with the same absolute value, but with opposite signs. The constants can be defined in complete generality by $\pm(mc/\hbar)^2$, where m is at this point an arbitrary constant yet to be determined. One therefore has

$$\frac{\square_x \phi_1(x)}{\phi_1(x)} = -(mc/\hbar)^2, \quad (11)$$

and

$$\rho_0^{-2} \frac{\square_v \phi_2(v)}{\phi_2(v)} = (mc/\hbar)^2. \quad (12)$$

Possible solutions to Eqs. (11) and (12) are given by

$$\phi_1^\pm(x) = \phi_{10} e^{\pm i k x}, \quad (13)$$

and

$$\phi_2^\pm(v) = \phi_{20} e^{\mp q v}, \quad (14)$$

where ϕ_{10} and ϕ_{20} are constants, $kx \equiv k_\mu x^\mu$, $qv \equiv q_\mu v^\mu$, and k^μ and q^μ are Lorentz four-vectors, still to be determined. For $\phi_2^\pm(v)$ in Eq. (14) to be bounded for $|qv| \rightarrow \infty$, and the exponential to be decreasing, the negative sign must be chosen in the exponent for $qv > 0$, and the positive sign must be chosen for $qv < 0$. Next, substituting Eqs. (13) and (14) in Eqs. (11) and (12), respectively, one obtains

$$k^2 = (mc/\hbar)^2 \quad (15)$$

and

$$\rho_0^{-2} q^2 = (mc/\hbar)^2. \quad (16)$$

(In obtaining Eq. (16), we temporarily ignore the constraint on the sign of the exponent in Eq. (14).) Next, Eq. (11) can be rewritten as follows:

$$\hbar^2 \square_x \phi_1(x) + m^2 c^2 \phi_1(x) = 0, \quad (17)$$

and has the same form as the ordinary massive Klein-Gordon wave equation for free scalar particles of mass m [12]. The parameter m can therefore be interpreted as the free-particle mass m . Also, for the particle four-momentum p^μ , one has

$$p^2 = m^2 c^2, \quad (18)$$

where $p^2 = p_\mu p^\mu$. No a priori assumption is made here regarding the explicit relationship between the four-velocity coordinate v^μ and the particle momentum p^μ . However, the quantum mechanical relationship between the particle four-momentum and the associated wave frequency and wavelength is assumed to be maintained. If we compare Eq. (15) (which arises from Eqs. (11) and (13)) with Eq. (18), we obtain (most simply)

$$k = \frac{p}{\hbar}. \quad (19)$$

We also assume, based on a comparison of Eq. (16) with Eq. (18), that (most simply)

$$q = \frac{\rho_0 p}{\hbar}. \quad (20)$$

It is evident that other possibilities exist, such as

$$q^\mu = (\rho_0/\hbar) l^\mu_\nu p^\nu, \quad (21)$$

where l^μ_ν is some linear transformation satisfying

$$l^\alpha_\mu l^\beta_\nu \eta_{\alpha\beta} = \eta_{\mu\nu}. \quad (22)$$

This and other possibilities remain to be explored. However, for simplicity, I assume Eq. (20) in the present work.

Next, substituting Eqs. (19) and (20) in Eqs. (13) and (14), and explicitly enforcing boundedness, we obtain

$$\phi_1^\pm(x) = \phi_{10} e^{\pm i p x / \hbar}, \quad (23)$$

$$\phi_2^\pm(v) = \phi_{20} e^{\mp \rho_0 p v / \hbar} \theta(\pm \rho_0 p v / \hbar), \quad (24)$$

where $p x = p_\mu x^\mu$, $p v = p_\mu v^\mu$, and $\theta(x)$ is the Heaviside step function,

$$\theta(x) = \begin{cases} 1 & , \quad x > 0 \\ 1/2 & , \quad x = 0 \\ 0 & , \quad x < 0 \end{cases}. \quad (25)$$

The appearance of the step function in Eq. (24) ensures that the correct sign choices in the exponent are implemented so that the exponential is decreasing for increasing values of the magnitude of its exponent, and the field remains bounded. One must, however, demonstrate that Eq. (12) is still satisfied when the step function is included in Eq. (24). To see that Eq. (12) is satisfied, I substitute Eq. (24) in Eq. (12), and obtain

$$\square_v \phi_2^\pm(v) = \phi_{20} \left(\frac{\rho_0 m c}{\hbar} \right)^2 [\delta'(\pm \rho_0 p v / \hbar) - 2\delta(\pm \rho_0 p v / \hbar) + \theta(\pm \rho_0 p v / \hbar)] e^{\mp \rho_0 p v / \hbar}, \quad (26)$$

where $\delta(x)$ is the Dirac delta function and $\delta'(x) \equiv \frac{d}{dx} \delta(x)$. I proceed to demonstrate that the two delta-function terms do not contribute because their arguments are nonvanishing and finite. First, the free-particle energy E must always be positive. Also, the four-vector momentum p^μ in Minkowski spacetime is given by

$$p^\mu = \left(\frac{E}{c}, \vec{p} \right) = \left(\left[m^2 c^2 + |\vec{p}|^2 \right]^{1/2}, \vec{p} \right), \quad (27)$$

where \vec{p} is the particle three-momentum. The four-velocity v^μ , which parameterizes the fiber of the spacetime tangent bundle, must lie on the four-velocity shell:

$$v^2 = g_{\mu\nu}v^\mu v^\nu = g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \frac{g_{\mu\nu} dx^\mu dx^\nu}{ds^2} = 1, \quad (28)$$

since the spacetime interval ds satisfies

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (29)$$

Therefore, in the flat Minkowski spacetime, Eq. (4), one has

$$v^2 = \eta_{\mu\nu}v^\mu v^\nu = (v^0)^2 - \left| \frac{d\vec{x}}{ds} \right|^2 = 1, \quad (30)$$

and consequently, for ordinary velocity $d\vec{x}/dt$, it follows that

$$v^0 = \pm \left(1 + \left(\frac{dt}{ds} \right)^2 \left| \frac{d\vec{x}}{dt} \right|^2 \right)^{1/2}, \quad (31)$$

$$\left(\frac{ds}{dt} \right)^2 = \frac{(c^2 dt^2 - |d\vec{x}|^2)}{dt^2} = \left(\frac{c}{\gamma} \right)^2, \quad (32)$$

where

$$\gamma = \left(1 - \frac{|d\vec{x}/dt|^2}{c^2} \right)^{-1/2}. \quad (33)$$

Therefore, substituting Eq. (32) in Eq. (31), and using Eq. (33), one obtains for the time component of four-velocity,

$$v^0 = \pm \gamma. \quad (34)$$

For classical particles and fields, four-velocity v^μ is future directed, namely,

$$v^\mu \equiv \left(\gamma, \gamma \frac{d\vec{x}/dt}{c} \right), \quad (35)$$

corresponding to the positive sign in Eq. (34). However, for generality, we retain both signs in Eq. (34). For quantum fields, we shall see that it is necessary to consider both future and past directed four-velocities, corresponding to the positive and negative signs, respectively, in Eq. (34). Also, one has

$$\vec{v} = \frac{d\vec{x}}{ds} = \frac{dt}{ds} \frac{d\vec{x}}{dt} = \frac{v^0}{c} \frac{d\vec{x}}{dt}, \quad (36)$$

or using Eq. (34), one obtains for the spatial component of four-velocity:

$$\vec{v} = \pm \gamma \frac{d\vec{x}/dt}{c}. \quad (37)$$

Combining Eqs. (34) and (37) produces

$$v^\mu = \pm \left(\gamma, \gamma \frac{d\vec{x}/dt}{c} \right). \quad (38)$$

Next, using Eqs. (27) and (38), one obtains

$$pv = \pm \gamma \left(\left(m^2 c^2 + |\vec{p}|^2 \right)^{1/2} - \frac{\vec{p} \cdot d\vec{x}/dt}{c} \right), \quad (39)$$

or equivalently,

$$pv = \pm mc (1 - \beta^2)^{-1/2} F \left(\frac{|\vec{p}|}{mc} \right). \quad (40)$$

Here, the function $F(x)$ is defined by

$$F(x) = (1 + x^2)^{1/2} - (\beta \cos \theta')x, \quad (41)$$

where

$$\beta = |d\vec{x}/dt|/c < 1, \quad (42)$$

and

$$\theta' = \cos^{-1} \left(\frac{\vec{p} \cdot d\vec{x}/dt}{|\vec{p}| |d\vec{x}/dt|} \right). \quad (43)$$

The function $F(x)$, Eq. (41), has a minimum at $x = x_{\min}$ for

$$0 = \partial F / \partial x = x(1 + x^2)^{-1/2} - \beta \cos \theta', \quad (44)$$

or

$$x_{\min} = \beta \cos \theta' (1 - \beta^2 \cos^2 \theta')^{-1/2}. \quad (45)$$

Substituting Eq. (45) in Eq. (41), one has

$$F(x_{\min}) = (1 - \beta^2 \cos^2 \theta')^{1/2}. \quad (46)$$

Therefore, if one substitutes Eq. (46) in Eq. (40), and uses Eq. (42), one obtains the following inequalities:

$$\begin{aligned} pv &> +mc [(1 - \beta^2 \cos^2 \theta') / (1 - \beta^2)]^{1/2} > +mc \\ &< -mc [(1 - \beta^2 \cos^2 \theta') / (1 - \beta^2)]^{1/2} < -mc. \end{aligned} \quad (47)$$

Therefore, for nonvanishing m , it is true that

$$\delta (\pm \rho_0 pv / \hbar) = 0, \quad (48)$$

and

$$\delta' (\pm \rho_0 pv / \hbar) = 0. \quad (49)$$

Substituting Eqs. (48) and (49) in Eq. (26), and using Eq. (24), one obtains

$$\square_v \phi_2^\pm(v) = \left(\frac{\rho_0 mc}{\hbar} \right)^2 \phi_2^\pm(v). \quad (50)$$

Comparing Eq. (50) with Eq. (12), we can conclude that $\phi_2^\pm(v)$, given by Eq. (24), does in fact satisfy Eq. (12).

Next, substituting Eqs. (23) and (24) in Eq. (9), we obtain the following positive and negative frequency mode solutions, $\phi_p^\pm(x, v)$, to Eq. (6):

$$\phi_p^\pm(x, v) = \phi_0 e^{\mp i p x / \hbar} e^{\mp \rho_0 p v / \hbar} \theta (\pm \rho_0 p v / \hbar), \quad (51)$$

where ϕ_0 is a constant, and p^0 is positive and given by

$$p^0 = \left(m^2 c^2 + |\vec{p}|^2 \right)^{1/2}. \quad (52)$$

III. SCALAR QUANTUM FIELD

One obtains the corresponding free relativistic scalar quantum field $\phi(x, v)$ by integrating over the ordinary three-momentum of the invariant positive and negative frequency modes $\phi_p^\pm(x, v)$ of Eq. (51), including appropriate creation and annihilation operators [12]. It follows from Eqs. (34), (38), (39), (47), (51) and the property of the Heaviside step function, Eq. (25), that for particles (positive frequency) and antiparticles (negative frequency), respectively, nonvanishing support is provided for $v^0 = +\gamma$ and for $v^0 = -\gamma$, respectively. Thus pv is effectively positive for a particle and negative for an antiparticle. Hence, the free relativistic scalar quantum field $\phi(x, v)$ in the spacetime tangent bundle is given by [1–4]

$$\phi(x, v) = 2 \int \frac{d^3 p}{(2\pi)^{3/2} (2p^0)^{1/2}} \left[e^{-ipx/\hbar} e^{-\rho_0 pv/\hbar} \theta(\rho_0 pv/\hbar) a(p) + e^{ipx/\hbar} e^{\rho_0 pv/\hbar} \theta(-\rho_0 pv/\hbar) a^\dagger(p) \right], \quad (53)$$

in which $a(p)$ and $a^\dagger(p)$ are the annihilation and creation operators, respectively, for particles and their uncharged antiparticles, and the operators satisfy the usual commutation relations [12]:

$$[a(p), a^\dagger(p')] = \delta^3(p - p'), \quad (54)$$

$$[a(p), a(p')] = 0, \quad (55)$$

$$[a^\dagger(p), a^\dagger(p')] = 0. \quad (56)$$

To accommodate normalization associated with the four-velocity coordinate of the spacetime tangent bundle, the integration measure in the tangent space must be appropriately chosen. This would be important, for example, in calculating the particle number density, or the average particle four-velocity. Also, it is important to emphasize that throughout the present work, the field describes a neutral particle, which is its own antiparticle. In Eq. (53) I choose the field normalization to agree with that for a canonical free relativistic scalar quantum field [12] in the limit of infinite maximal proper acceleration a_0 or, equivalently, vanishing ρ_0 . Thus, using Eqs. (25) and (53), we can see that

$$\phi(x, v) \xrightarrow{\rho_0 \rightarrow 0} \int \frac{d^3 p}{(2\pi)^{3/2} (2p^0)^{1/2}} \left[e^{-ipx/\hbar} a(p) + e^{ipx/\hbar} a^\dagger(p) \right], \quad (57)$$

which is the appropriate limit for canonical relativistic quantum field theory.

In the regions of nonvanishing support, both terms of Eq. (53) effectively have the decreasing exponential factor $\exp(-\rho_0 |pv|/\hbar)$, which can be rewritten using Eqs. (3) and (39) as

$$\exp(-\rho_0 |pv|/\hbar) = \exp \left(-\frac{1}{2\pi\alpha} \frac{\gamma m}{m_{pl}} \left\{ \left[1 + \left(\frac{|\vec{p}|}{mc} \right)^2 \right]^{1/2} - \frac{\vec{p} \cdot \vec{dx}/dt}{mc^2} \right\} \right), \quad (58)$$

where the Planck mass m_{pl} is given by

$$m_{pl} = \left(\frac{\hbar c}{G} \right)^{1/2}. \quad (59)$$

The spectrum is exponentially cut off beyond the Planck mass, resulting in an automatic field regularization.

IV. MICROCAUSALITY

Next I address the subject of microcausality [12–21] for Lorentz-invariant quantum fields in the spacetime tangent bundle with a Minkowski spacetime base manifold, by calculating the appropriate field commutators [5, 22]. I proceed to argue that, provided the adjoint operation on the fields is consistently generalized to include four-velocity sign-inversion, the necessary commutators are vanishing, and the field is microcausal. There are, however, Planck-scale modifications of the causal domain, that only become significant for extremely large relative four-velocities at the separated spacetime points. For vanishing relative four-velocities, the causal domain is canonical. The generalized

adjoint operation is to be consistent with the facts that (1) the time-component of four-velocity is positive for particles and negative for antiparticles, (2) the adjoint operation on the scalar field corresponds to charge conjugation, which replaces particles by antiparticles and antiparticles by particles, and (3) the neutral scalar particle is its own antiparticle.

Using Eq. (53), it can be shown that outside the causal domain, one has [5]

$$[\phi(x, v), \phi(x', v')] = \frac{m\bar{c}\hbar}{\pi^2} [\theta(\rho_0 m c v^0 / \hbar) \theta(-\rho_0 m c v^{0'} / \hbar) - \theta(\rho_0 m c v^{0'} / \hbar) \theta(-\rho_0 m c v^0 / \hbar)] \times \frac{K_1\left(\frac{m\bar{c}}{\hbar} [-(x-x'-i\rho_0(v-v'))^2]^{1/2}\right)}{[-(x-x'-i\rho_0(v-v'))^2]^{1/2}}, \quad (60)$$

where K_1 is the modified Bessel function of the third kind and of order 1 [23]. Alternatively, using Eqs. (53), one also obtains [5]

$$[\phi(x, v), \phi(x', v')] = 4 \left[\theta(\rho_0 m c v^0 / \hbar) \theta(-\rho_0 m c v^{0'} / \hbar) \int \frac{d^3 p}{(2\pi)^3 (2p^0)} e^{-ip(x-x')/\hbar} e^{-\rho_0 p(v-v')/\hbar} - \theta(\rho_0 m c v^{0'} / \hbar) \theta(-\rho_0 m c v^0 / \hbar) \int \frac{d^3 p}{(2\pi)^3 (2p^0)} e^{ip(x-x')/\hbar} e^{-\rho_0 p(v'-v)/\hbar} \right], \quad (61)$$

in which the integrals remain unevaluated. Equation (61) can be used for those cases in which the evaluation of the commutator depends only on the properties of the Heaviside step function, Eq. (25), and in which the arguments leading to Eq. (60) may not apply.

Here, except for possible Planck-scale effects, spacelike spacetime separations are of primary interest, since micro-causality is the issue. It is well to note explicitly that the modified Bessel function, and also the reciprocal of the denominator, in Eq. (60) both become singular when the Bessel-function argument is vanishing, namely when [5, 22]

$$[x - x' - i\rho_0(v - v')]^2 = 0. \quad (62)$$

Equivalently, one has

$$(x - x')^2 = \rho_0^2 (v - v')^2, \quad (63)$$

and

$$\rho_0(v - v') \cdot (x - x') = 0. \quad (64)$$

Thus, if the relative four-velocity is vanishing, the modified Bessel function is singular on the light cone, $(x - x')^2 = 0$, as in the canonical theory. Equations (62)–(64) exhibit, explicitly, the Planck-scale effects on the boundary of the causal domain. Note that equal-times are a special case of spacelike separations, $(x - x')^2 < 0$, and, in accordance with special relativity, it is clearly true that at equal times, the ordinary velocities at the separated spacetime points must be the same, in order to maintain simultaneity, and this is consistent with the corresponding four-velocities being equal in Eq. (63). Just as the light cone, $(x - x')^2 = 0$, separates the causal timelike domain from the acausal spacelike domain for a canonical relativistic field in Minkowski spacetime, so too, the six-dimensional hypersurface defined by Eqs. (63) and (64) separates the causal domain from the acausal domain for the field $\phi(x, v)$ in the Minkowski-spacetime tangent bundle.

Equations (63) and (64) can also be written as

$$(\Delta x^0)^2 = \left| \Delta \vec{x} \right|^2 + \rho_0^2 (\Delta v^0)^2 - \rho_0^2 \left| \Delta \vec{v} \right|^2, \quad (65)$$

and

$$\rho_0 \Delta v^0 \Delta x^0 = \rho_0 \Delta \vec{v} \cdot \Delta \vec{x}, \quad (66)$$

respectively, where the components of spacetime separation are $\Delta x^0 = x^0 - x^{0'}$, $\Delta \vec{x} = \vec{x} - \vec{x}'$, and the components of relative four-velocity are $\Delta v^0 = v^0 - v^{0'}$, $\Delta \vec{v} = \vec{v} - \vec{v}'$. Next, multiplying Eq. (65) by $(\Delta x^0)^2$ and substituting Eq. (66) into the result, one obtains

$$(\Delta x^0)^4 - \left(\left| \Delta \vec{x} \right|^2 - \rho_0^2 \left| \Delta \vec{v} \right|^2 \right) (\Delta x^0)^2 - \rho_0^2 \left| \Delta \vec{v} \cdot \Delta \vec{x} \right|^2 = 0. \quad (67)$$

According to the quadratic formula, it then follows from Eq. (67) that

$$\Delta x^0 = \pm \left(\left| \Delta \vec{x} \right|^2 - \rho_0^2 \left| \Delta \vec{v} \right|^2 \right)^{1/2} \left[\frac{1}{2} \pm \frac{1}{2} \left(1 + \left(\frac{2\rho_0 \Delta \vec{v} \cdot \Delta \vec{x}}{\left| \Delta \vec{x} \right|^2 - \rho_0^2 \left| \Delta \vec{v} \right|^2} \right)^2 \right)^{1/2} \right]^{1/2}. \quad (68)$$

Choosing the positive sign inside the bracket and preceding the radical, so that Δx^0 is real, Eq. (68) can also be written as follows:

$$T = \pm \left\{ \frac{1}{2} (X^2 - 1) + \frac{1}{2} [X^4 + 2(2 \cos^2 \theta - 1) X^2 + 1]^{1/2} \right\}^{1/2}, \quad (69)$$

where

$$T \equiv \frac{\Delta x^0}{\rho_0 \left| \Delta \vec{v} \right|}, \quad (70)$$

$$X \equiv \frac{\left| \Delta \vec{x} \right|}{\rho_0 \left| \Delta \vec{v} \right|}, \quad (71)$$

and

$$\theta \equiv \cos^{-1} \left(\frac{\Delta \vec{v} \cdot \Delta \vec{x}}{\left| \Delta \vec{v} \right| \left| \Delta \vec{x} \right|} \right) \quad (72)$$

is the angle between the spatial component of relative four-velocity and the spatial-separation vector. Also, Eq. (66) can be written as

$$V = \frac{X}{T} \cos \theta, \quad (73)$$

where

$$V \equiv \frac{\rho_0 \Delta v^0}{\rho_0 \left| \Delta \vec{v} \right|}. \quad (74)$$

Substituting Eq. (69) in Eq. (73), one obtains

$$V = \pm X \cos \theta \left\{ \frac{1}{2} (X^2 - 1) + \frac{1}{2} [X^4 + 2(2 \cos^2 \theta - 1) X^2 + 1]^{1/2} \right\}^{-1/2}. \quad (75)$$

Also, using Eqs. (69) and (73), one has

$$V = \pm \left(\frac{1 + T^2}{1 + T^2 \sec^2 \theta} \right)^{1/2}. \quad (76)$$

Equations (69), (75) and (76) are plotted in Figs. 1, 2, and 3, respectively, for several values of θ . The figures correspond to representative slices of the six-dimensional hypersurface which is the boundary, Eqs. (63) and (64), of the causal domain [22].

Observe in Fig. 1 that for a large spatial component of relative four-velocity $\left| \Delta \vec{v} \right|$ (see Eqs. (69)–(71)), the Planck-scale effects [5] become manifest, and the causal domain penetrates significantly into the canonical spacelike region for spatial separations within several multiples of $\rho_0 \left| \Delta \vec{v} \right|$. The effect is least when the spatial part of the relative four-velocity $\Delta \vec{v}$ is nearly parallel or antiparallel to the spatial-separation vector $\Delta \vec{x}$, corresponding to $\theta \approx 0$ or π .

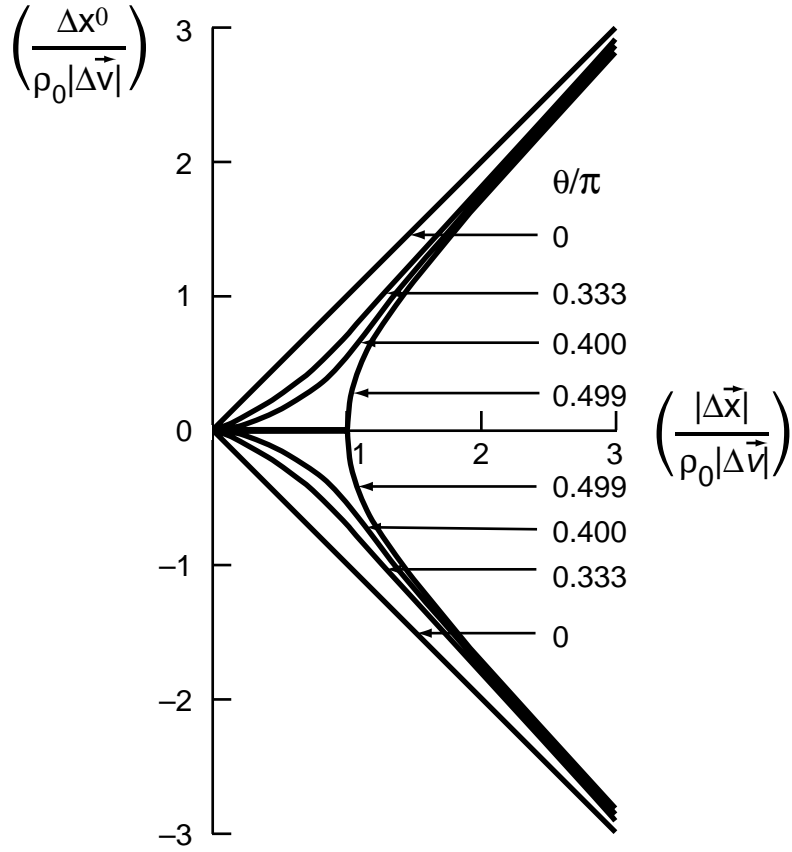


FIG. 1: Causal boundary Eq. (69): $(\Delta x^0/\rho_0|\Delta\vec{v}|)$ as a function of $(|\Delta\vec{x}|/\rho_0|\Delta\vec{v}|)$ for various values of the angle θ between the spatial component $\Delta\vec{v}$ of relative four-velocity and the spatial-separation vector $\Delta\vec{x}$.

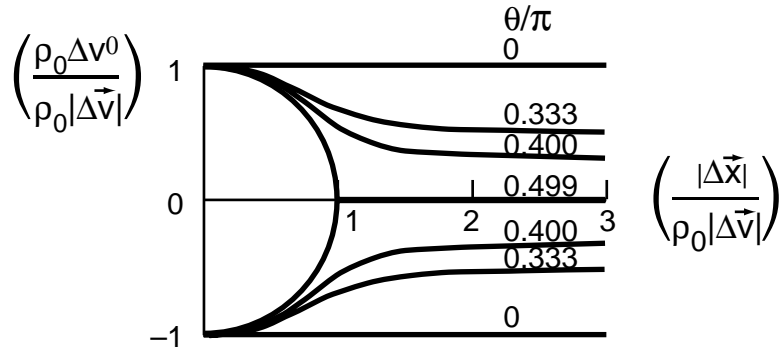


FIG. 2: Causal boundary Eq. (75): $(\rho_0\Delta v^0/\rho_0|\Delta\vec{v}|)$ as a function of $(|\Delta\vec{x}|/\rho_0|\Delta\vec{v}|)$ for various values of the angle θ between the spatial component $\Delta\vec{v}$ of relative four-velocity and the spatial-separation vector $\Delta\vec{x}$.

The effect is greatest when $\theta = \pi/2$ and the spatial part of the relative four-velocity is perpendicular to the spatial-separation vector. For $\theta = \pi/2$ in Eq. (69), T is complex for $X < 1$, and the causal boundary is on the hyperbola $T = \pm (X^2 - 1)^{1/2}$, which corresponds to the single-sheeted 4-hyperboloid [24] in spacetime:

$$(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 - (\Delta x^0)^2 = \rho_0^2 |\Delta\vec{v}|^2. \quad (77)$$

From Eq. (69), it follows that

$$\frac{dT}{dX}|_{X=0} = \pm \cos \theta. \quad (78)$$

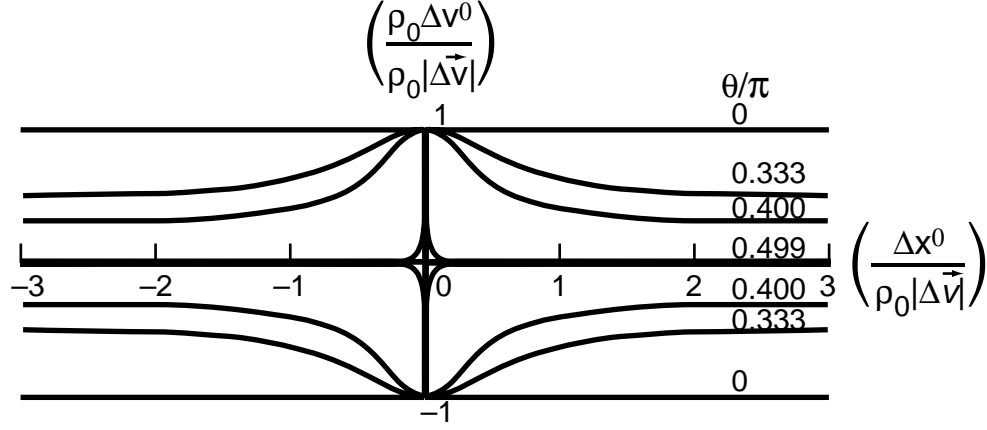


FIG. 3: Causal boundary Eq. (76): $(\rho_0 \Delta v^0 / \rho_0 |\Delta \vec{v}|)$ as a function of $(\Delta x^0 / \rho_0 |\Delta \vec{v}|)$ for various values of the angle θ between the spatial component $\Delta \vec{v}$ of relative four-velocity and the spatial-separation vector $\Delta \vec{x}$.

Thus in Fig. 1, the pointed section of the hypersurface near the singular point at the origin becomes sharper as θ ranges from 0 to $\pi/2$. In the limit as $\theta \rightarrow \pi/2$, it becomes an infinitesimally thin cusp, tangent (above and below) to the segment $0 \leq X \leq 1$ of the X -axis at $T = 0$. The rest of the causal boundary for $\theta \rightarrow \pi/2$ becomes tangent to the hyperbola $T = \pm (X^2 - 1)^{1/2}$. The segment corresponds to the simultaneous $(\Delta x^0 = 0)$ and spacelike 3-sphere $(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = \rho_0^2 |\Delta \vec{v}|^2$ and its interior, and the hyperbola corresponds to the 4-hyperboloid Eq. (77) [24]. All points on the tangential infinitesimally-thin cusp (as well as those on the tangent to the hyperbola) lie on the hypersurface on which the field commutator, Eq. (60), is singular [5], and thus correspond to causal spacetime separations (even though the points of the 3-sphere and its interior are simultaneous). The geometry indicates that at the Planck scale, causal connectivity of the fields may occur between spacelike separated points, and also at larger scales for extremely large relative four-velocities. This suggests the possibility that for nominal values of the spatial component of relative four-velocity, causally-connected Planck-size spatially extended excitations (such as strings or branes) may not be incompatible with the structure of the causal domain near the Planck scale. The spatial extent of the distortions of the canonical causal domain scales linearly with the magnitude of the spatial component of relative four-velocity, and may extend well beyond the Planck scale for extremely large relative four-velocities. However, extremely large relative four-velocities also require extremely large four-velocities, for which the field is exponentially cut off as in Eq. (58).

Proceeding with the evaluation of Eq. (60) in the limit of infinite maximal proper acceleration, using Eqs. (3) and (25), one gets

$$[\phi(x, v), \phi(x', v')] \xrightarrow{\rho_0 \rightarrow 0} 0, \quad (79)$$

as must be the case for consistency with the canonical scalar field [12], which satisfies microcausality. Because of the Heaviside step functions in Eq. (61), one can immediately see, using Eqs. (25) and (38) that

$$[\phi(x, v), \phi(x', v')] \Big|_{v^0 v'^0 > 0} = 0. \quad (80)$$

Equation (80) is consistent with microcausality. That Eq. (80) also applies when the fields have four-velocities that are both past-directed need not be a priori inconsistent with relativistic causality.

If v is future directed ($v^0 > 0$) and v' is past directed ($v'^0 < 0$), one obtains, using Eqs. (25) and (38) in Eq. (60), the nonvanishing value:

$$[\phi(x, v), \phi(x', v')] \Big|_{v^0 > 0, v'^0 < 0} = \frac{m\hbar}{\pi^2} \frac{K_1 \left(\frac{mc}{\hbar} \left[-(x - x' - i\rho_0(v - v'))^2 \right]^{1/2} \right)}{\left[-(x - x' - i\rho_0(v - v'))^2 \right]^{1/2}}. \quad (81)$$

Equation (81) is apparently not inconsistent with microcausality because the four-velocities at the separated spacetime points are oppositely directed in time, in which case the concept of causality loses any obvious meaning. Also, the

associated four-velocity shells are disjoint. The spacetime points are however causally separated on the singular hypersurface on which the commutator becomes singular [5]. Also, note that, because the modified Bessel function K_1 drops off exponentially for large arguments [25], the right side of Eq. (81) will typically (except for extremely large relative four-velocities) be very small for space-like separations exceeding several Compton wavelengths, \hbar/mc . Because ρ_0 is only of the order of the Planck length, the effect of the imaginary parts in the denominator and in the argument of the modified Bessel function can be expected to be negligible, except at extremely large separations in the four-velocity tangent space of the spacetime tangent bundle, for which Planck-scale structure becomes manifest. Analogously to the derivation of Eq. (81), one obtains

$$[\phi(x, v), \phi(x', v')] \Big|_{v^0 < 0, v^{0'} > 0} = -\frac{m\hbar c}{\pi^2} \frac{K_1 \left(\frac{mc}{\hbar} \left[-(x - x' - i\rho_0(v - v'))^2 \right]^{1/2} \right)}{\left[-(x - x' - i\rho_0(v - v'))^2 \right]^{1/2}}, \quad (82)$$

which is also apparently not inconsistent with microcausality, for the same reasons already given for Eq. (81).

Next note that, without any qualifications on the canonical adjoint operator, the field ϕ of Eq. (53) is apparently not Hermitian. It is important, for the consistent interpretation of the commutators involving the ordinary adjoint field $\phi^\dagger(x, v)$ [5], to introduce a generalized adjoint field $(\phi(x, v))^\dagger$ as the charge conjugate field obtained by particle-antiparticle conjugation, analogously to the canonical case [12], with particles replaced by antiparticles, and antiparticles replaced by particles, and also to recall that particles and antiparticles only have nonvanishing support for $v^0 > 0$ and $v^0 < 0$, respectively in Eq. (53). Although charge conjugation has nothing to do with the space or time coordinates (as in the canonical theory), it apparently does have to do with the tangent space coordinates of the spacetime tangent bundle, because particles have nonvanishing support for future-directed four-velocities, while antiparticles have nonvanishing support for past-directed four-velocities [1–4]. It is thus logical to define the charge-conjugate field $\phi_c(x, v)$ and the generalized adjoint $(\phi(x, v))^\dagger$ of the quantum field $\phi(x, v)$ by [5]

$$\phi_c(x, v) = (\phi(x, v))^\dagger \equiv \phi^\dagger(x, -v). \quad (83)$$

Equation (83) is compatible with the canonical adjoint operation [12], since, in the latter, the fields are independent of four-velocity, and therefore four-velocity does not even enter into consideration. Next, substituting Eq. (53) in the second equality of Eq. (83), it follows that Hermiticity is upheld in a generalized sense, namely,

$$(\phi(x, v))^\dagger \equiv \phi(x, v). \quad (84)$$

Equations (83) and (84), analogously to the canonical case [12], guarantee the identity of the particle and antiparticle states for the neutral scalar quantum field, namely,

$$\phi_c(x, v) = \phi(x, v). \quad (85)$$

Equation (85) is a consistent generalization of the corresponding canonical relation [12]. In the latter, of course, four-velocity does not enter. One is however reminded of the Stückelberg-Feynman idea that an antiparticle can be interpreted as a particle going backward in time [26–28].

Note also that by using Eqs. (53), (83) and (84), one can show that [5]

$$\begin{aligned} \left[\phi(x, v), (\phi(x', v'))^\dagger \right] &= \frac{m\hbar c}{\pi^2} \left[\theta(\rho_0 m c v^0 / \hbar) \theta(-\rho_0 m c v^{0'} / \hbar) - \theta(-\rho_0 m c v^0 / \hbar) \theta(\rho_0 m c v^{0'} / \hbar) \right] \\ &\times \frac{K_1 \left(\frac{mc}{\hbar} \left[-(x - x' - i\rho_0(v - v'))^2 \right]^{1/2} \right)}{\left[-(x - x' - i\rho_0(v - v'))^2 \right]^{1/2}}. \end{aligned} \quad (86)$$

With the generalized adjoint operation, the appropriate field commutators are all vanishing and are consistent with microcausality [5]. Planck-scale structure does manifests itself in modifications of the boundary of the causal domain, but significant modifications appear only for extremely large relative four-velocities at the separated spacetime points.

Since the fields themselves may not correspond to physical observables, it is important to consider bilinear forms of the fields, corresponding to local observables [13]. Microcausality requires that observables commute for spacelike separations in spacetime. A possible operator, corresponding to a local observable, is defined by

$$O(x, v) \equiv (\phi(x, v))^\dagger \phi(x, v), \quad (87)$$

or, substituting Eq. (83) in Eq. (87), one has, for the observable, the following expression:

$$O(x, v) = \phi^\dagger(x, -v) \phi(x, v). \quad (88)$$

Using Eqs. (88), (83), and (87), one obtains

$$O^\dagger(x, v) \equiv (\phi(x, v))^\dagger (\phi^\dagger(x, -v))^\dagger = \phi^\dagger(x, -v)\phi(x, v) = O(x, v). \quad (89)$$

Thus, the observable is Hermitian in the generalized sense.

Next consider the commutator $[O(x, v), O(x', v')]$, which must be vanishing for spacelike separations in spacetime, and future-directed four-velocities, if consistency with microcausality is to be upheld. First, using Eq. (87), one has

$$[O(x, v), O(x', v')] = [(\phi(x, v))^\dagger \phi(x, v), (\phi(x', v'))^\dagger \phi(x', v')]. \quad (90)$$

Then substituting Eq. (83) in Eq. (90), one obtains

$$[O(x, v), O(x', v')] = [\phi^\dagger(x, -v)\phi(x, v), \phi^\dagger(x', -v')\phi(x', v')]. \quad (91)$$

If one expands Eq. (91), it becomes

$$\begin{aligned} [O(x, v), O(x', v')] &= \phi^\dagger(x, -v)\phi^\dagger(x', -v') [\phi(x, v), \phi(x', v')] \\ &+ \phi^\dagger(x, -v) [\phi(x, v), \phi^\dagger(x', -v')] \phi(x', v') + \phi^\dagger(x', -v') [\phi^\dagger(x, -v), \phi(x', v')] \phi(x, v) \\ &+ [\phi^\dagger(x, -v), \phi^\dagger(x', -v')] \phi(x', v')\phi(x, v). \end{aligned} \quad (92)$$

Next, using the appropriate field commutators [5] in Eq. (92), one obtains

$$[O(x, v), O(x', v')] = 0. \quad (93)$$

Equation (93) is consistent with microcausality. However, extremely large relative four-velocities will give rise to significant Planck-scale modifications of the causal domain, Eqs. (63) and (64).

V. CONCLUSIONS

A theory of a Lorentz-invariant quantum field in the spacetime tangent bundle has been formulated, based on maximal-acceleration invariance. The field was demonstrated to have a physically based Planck-scale effective regularization with an effective spectral cutoff near the Planck mass. Also it was argued that, provided the adjoint field is consistently generalized, the necessary commutators are vanishing, and the field is microcausal. There are, however, Planck-scale modifications of the causal domain, but they only become significant for extremely large relative four-velocities at the separated spacetime points. The causal domain is canonical for vanishing relative four-velocities. The geometry of the causal domain suggests that at the Planck scale, causal connectivity may occur between spacelike separated points, and also at larger scales for extremely large relative four-velocities.

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