Relativistic Wigner Function, Charge Variable and Structure of Position Operator

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Abstract

We consider the relativistic Wigner representation for a class of observables which are arbitrary combinations of the standard coordinate and momentum operators (independent of the charge variable). One should take into account the fact that eigenfunctions of the standard operator of coordinate contain both charge components, although the real physical states are not charge-violating. The evolution equation does not depend on this peculiarity under conditions when particle pairs creation is impossible. However, it leads to a specific constraint on the class of initial conditions and to effective increase of the coherence between eigenstates of Hamiltonian.

I. INTRODUCTION

Absence of a well-defined position operator in relativistic quantum mechanics is one of the features that lead to difficulties in interpretation. It is well-known, that standard operator of coordinate contains the odd part mixing the states with different signs of charge [1,2]. As a result, the eigenfunction of this operator contains both of charge components, what is prohibited by the charge superselection rule [3,4]. In their work [1], T.D. Newton and E.P. Wigner have introduced another operator of coordinate, that is known as Newton–Wigner position operator.

In the approach of H. Feshbach and F. Villars [2], this operator can be presented from the following suggestions. Hamiltonian of a free scalar charged particle is written as follows:

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$$\hat{H} = (\tau_3 + i\tau_2)\frac{\hat{p}^2}{2m} + \tau_3 mc^2. \tag{1}$$

In the Feshbach–Villars representation this Hamiltonian has the diagonal form:

$$\hat{H} = \tau_3 E(\hat{p}), \tag{2}$$

where

$$E(p) = \sqrt{m^2c^4 + c^2p^2}. \tag{3}$$

The transformation between the standard and Feshbach–Villars representation is given by the matrix:

$$U(\hat{p}) = \frac{1}{2\sqrt{mc^2E(\hat{p})}}\left[(E(\hat{p}) + mc^2) + (E(\hat{p}) - mc^2)\tau_1\right]. \tag{4}$$

The standard operator of coordinate in the Feshbach–Villars representation contains even and odd parts:

$$\hat{q} = i\hbar\partial_p - i\frac{\hbar c^2 p}{2E^2(p)}\tau_1. \tag{5}$$

The even part of this operator

$$[\hat{q}] = i\hbar\partial_p \tag{6}$$

coincides with the Newton–Wigner position operator $\hat{\xi}$, and the operator of momentum in Hamiltonian (2) can be written through this operator in the canonical form:

$$\hat{p} = -i\hbar\partial_\xi. \tag{7}$$

One has more complicated situation in the case of a static magnetic field. Both coordinate and momentum operators have the odd parts here. Hamiltonian

$$\hat{H} = (\tau_3 + i\tau_2)\frac{(\hat{\mathbf{p}} - e\mathbf{A}(\hat{\mathbf{q}}))^2}{2m} + \tau_3 mc^2 \tag{8}$$

is transformed to the diagonal form

$$\hat{H} = \tau_3 E(\hat{n}) \tag{9}$$

with the following matrix:

$$U(\hat{n}) = \frac{1}{2\sqrt{mc^2E(\hat{n})}}\left[(E(\hat{n}) + mc^2) + (E(\hat{n}) - mc^2)\tau_1\right]. \tag{10}$$

$\hat{n}$ is "oscillator-like" operator in these equations. It has common eigenfunctions with Hamiltonian (8) and eigenvalues n (for the continuous part of the spectrum these numbers have not integer but real values). This operator can be expressed due to certain operators $\hat{\pi}$ and $\hat{\xi}$ in such a way that $E(\hat{n})$ in (9) takes the canonical form:

$$E(\hat{n}) = \sqrt{m^2c^4 + c^2\left(\hat{\pi} - e\mathbf{A}\left(\hat{\xi}\right)\right)^2}. \quad (11)$$

Generally speaking, $\hat{\pi}$ and $\hat{\xi}$ do not coincide with even parts of the corresponding operators in (8). This is the difference from the case of a free particle.

The theory with Hamiltonians (2, 9) (theory with non-local Hamiltonian or non-local theory [5,6]) does not have difficulties in the probability interpretation. However, if one accepts π and ξ as the real observable momentum and coordinate, the difficulties with the Lorentz-invariance appear. There exists a very interesting approach, where this feature is explained by the existence of the preferred reference frame in the Universe [7]. Nevertheless, the real operator of coordinate can do have nontrivial charge structure. It may reveal itself in the expected values of observables and demands some additional suggestions about the measurement of the relativistic coordinate.

Following [8], in this paper we consider the formalism of the Wigner function [9] in a constant magnetic field taking into account the nontrivial charge structure of the coordinate and momentum operators.

II. CHARGE-INVARIANT OBSERVABLES

According to [9], for a consistent development of the Wigner function formalism, one should proceed from the Weyl transformation:

$$\hat{A}_\alpha{}^\beta = \sum_{\gamma=\pm1} \int\limits_{-\infty}^{+\infty} A_\gamma{}^\beta(p,q)\,\hat{W}_\alpha{}^\gamma(p,q)\,dpdq, \quad (12)$$

where $\hat{W}_\alpha{}^\beta(p,q)$ is the operator of quasi-probability density, $\hat{A}_\alpha{}^\beta$ is the operator of an observable and $A_\alpha{}^\beta(p,q)$ is the corresponding matrix-valued Weyl symbol. Greek indices enumerate the component of charge space and take values ±1.

The momentum (or coordinate) part of the eigenfunction of the Hamiltonian (8,9) we represent as a solution of the following eigenvalues problem:

$$E(\hat{n})\,\varphi_n(p) = E(n)\,\varphi_n(p)\,. \quad (13)$$

Consider observables with the matrix-valued Weyl symbols being proportional to the identity matrix:

$$A_\alpha{}^\beta(p,q) = A(p,q)\delta_\alpha{}^\beta. \quad (14)$$

We denominate them as charge-invariant observables. The Weyl transformation for such observables in the energy representation has the following form:

$$A^E{}_{nm;\alpha}{}^\beta = R_\alpha{}^\beta(m,n)\int\limits_{-\infty}^{+\infty} A(p,q)W_{nm}(p,q)dpdq. \quad (15)$$

Here ${A^E}_{nm;\alpha}{}^{\beta}$ is the operator matrix of a charge-invariant observable in the energy representation, $W_{nm}\,(p,q)$ is the Hermitian generalization of the Wigner function

$$W_{nm}(p,q)=\frac{1}{(2\pi\hbar)^d}\int\limits_{-\infty}^{+\infty}\varphi_m^*\left(p+\frac{P}{2}\right)\varphi_n\left(p-\frac{P}{2}\right)\exp\left(-\frac{i}{\hbar}Pq\right)dP. \tag{16}$$

The main difference of the expression (15) from its analogues in the non-relativistic and non-local theories is in matrix-valued function ${R_\alpha}^{\beta}(m,n)$:

$${R_\alpha}^{\beta}(m,n)=\sum_{l,k=0}^{\infty}\sum_{\gamma=\pm1}U_{lm;\gamma}{}^{\beta}U^{-1}_{nk;\alpha}{}^{\gamma}=\varepsilon(m,n){\delta_\alpha}^{\beta}+\chi(m,n)\tau_{1;\alpha}{}^{\beta}, \tag{17}$$

where $\varepsilon(m,n)$ and $\chi(m,n)$ are written in the form:

$$\varepsilon(m,n)=\frac{E(m)+E(n)}{2\sqrt{E(m)E(n)}}, \tag{18}$$

$$\chi(m,n)=\frac{E(m)-E(n)}{2\sqrt{E(m)E(n)}}. \tag{19}$$

We denominate them as ε- and χ-factors.

Similar to [8], one can conclude that matrix elements of even and odd parts of the operator of an arbitrary charge-invariant observable are uniquely related to each other:

$$\{A^E\}_{nm;\alpha}{}^{\beta}=\frac{E(m)-E(n)}{E(m)+E(n)}\sum_{\gamma=\pm1}\tau_{1;\gamma}{}^{\beta}[A^E]_{nm;\alpha}{}^{\gamma}. \tag{20}$$

III. WIGNER FUNCTION AND QUANTUM LIOUVILLE EQUATION FOR CHARGE-INVARIANT OBSERVABLES

We introduce the Wigner function for the charge-invariant observables in such a way that their expected values are determined by the formula:

$$\bar{A}=\int\limits_{-\infty}^{+\infty}A(p,q)W(p,q)dpdq. \tag{21}$$

This object is the sum of four terms:

$$W(p,q)=\sum_{\alpha=\pm1}\left(W_{[\alpha]}(p,q)+W_{\{\alpha\}}(p,q)\right). \tag{22}$$

We denominate $W_{[\pm]}(p,q)$ as the even part of the Wigner function. It can be written as follows:

$$W_{[\pm]}(p,q) = \sum_{n,m=0}^{\infty} \varepsilon(m,n) W_{nm}(p,q) C^{*}_{m;\pm} C_{n;}{}^{\pm}. \tag{23}$$

In a similar way, we denominate $W_{\{\pm\}}(p,q)$ as the odd part of the Wigner function being determined as follows:

$$W_{\{\pm\}}(p,q) = \sum_{n,m=0}^{\infty} \chi(m,n) W_{nm}(p,q) C^{*}_{m;\pm} C_{n;}{}^{\mp}. \tag{24}$$

Here $C_{n;\alpha}$ is the wavefunction in the energy representation (the expansion coefficient of the wavefunction on eigenstates of the Hamiltonian (8)).

One can obtain evolution equations in a standard way, through applying well-known expressions for the Hermitian generalization of the Wigner function:

$$\partial_t W_{[\pm]}(p,q,t) = \pm \frac{2}{\hbar} E(p,q) \sin\left\{ \frac{\hbar}{2} \left(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q \right) \right\} W_{[\pm]}(p,q,t), \tag{25}$$

$$\partial_t W_{\{\pm\}}(p,q,t) = \pm \frac{2i}{\hbar} E(p,q) \cos\left\{ \frac{\hbar}{2} \left(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q \right) \right\} W_{\{\pm\}}(p,q,t). \tag{26}$$

Here we have introduced the following effective Hamiltonian:

$$E(p,q) = \sqrt[\star]{m^2 c^4 + c^2 (p - eA(q))^2}. \tag{27}$$

The square root is determined here by means of the star-product

$$\star \equiv e^{\frac{i\hbar}{2} \left(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q \right)}. \tag{28}$$

This is a specific feature, that distinguishes between relativistic and non-relativistic dynamics. In the non-relativistic case classical observables are brought into correspondence to the Weyl symbols. For the effective Hamiltonian in the relativistic case it is not true due to the determination of the square root. Note, that it is not related to the complicated charge structure of the coordinate and momentum operators.

Solutions of the system (25,26) are not independent, because of existence of specific constraits. Indeed, four components of the Wigner function are expressed via the two-component wavefunction. Therefore, one can combine them in such a way as to obtain the following equality:

$$\begin{array}{l} \int\limits_{-\infty}^{+\infty} \varepsilon^{-1}(p_1, p'; p_2, p'') W_{[+]}(\tfrac{1}{2}(p_1 + p_2), q_1) e^{\frac{i}{\hbar}(p_1 - p_2) q} dp_1 dp_2 dq \\ \times \int\limits_{-\infty}^{+\infty} \varepsilon^{-1}(p_1, p'; p_2, p'') W_{[-]}(\tfrac{1}{2}(p_1 + p_2), q_2) e^{\frac{i}{\hbar}(p_1 - p_2) q} dp_1 dp_2 dq \\ = \int\limits_{-\infty}^{+\infty} \chi^{-1}(p_1, p'; p_2, p'') W_{\{+\}}(\tfrac{1}{2}(p_1 + p_2), q_1) e^{\frac{i}{\hbar}(p_1 - p_2) q} dp_1 dp_2 dq \\ \times \int\limits_{-\infty}^{+\infty} \chi^{-1}(p_1, p'; p_2, p'') W_{\{-\}}(\tfrac{1}{2}(p_1 + p_2), q_2) e^{\frac{i}{\hbar}(p_1 - p_2) q} dp_1 dp_2 dq \end{array}. \tag{29}$$

Here the following generalized functions have been introduced:

$$\varepsilon^{-1}(p_1, p'; p_2, p'') = \sum_{nm} \varepsilon^{-1}(m,n)\varphi_m^*(p')\varphi_m(p_1)\varphi_n(p'')\varphi_n^*(p_2), \tag{30}$$

$$\chi^{-1}(p_1, p'; p_2, p'') = \sum_{nm} \chi^{-1}(m,n)\varphi_m^*(p')\varphi_m(p_1)\varphi_n(p'')\varphi_n^*(p_2). \tag{31}$$

In addition to this constraint, there exist expressions for complex conjugate values of the Wigner function, which can be considered as the specific constraints as well. To obtain these one should take the complex conjugation of (25, 26). After some simple transformations we obtain the following expressions:

$$W^*_{[\pm]}(p,q) = W_{[\pm]}(p,q), \tag{32}$$

$$W^*_{\{\pm\}}(p,q) = W_{\{\mp\}}(p,q). \tag{33}$$

The even components of the Wigner function are real. The odd components are complex conjugate to each other. Hence, their sum is real as well.

IV. PROPERTIES AND PHYSICAL SENSE

The evolution equations are the same in the both standard and non-local theories. However, the expected values of observables are in both cases different. To understand the reason of it, one needs to take into account that class of the functions on the phase space, which represent pure states, is restricted [10]. The standard theory distinguishes from the non-local one by the fact that the relevant Wigner functions representing real physical states belong to different classes.

There exist several criteria, that make it possible to select the Wigner function from the set of functions on the phase space. As an example, we consider one of the most known criteria, that is known as the quantization condition. This criterion results from the fact that Wigner function, as a function of two variables, is expressed via the wavefunction of one variable. In our case it can be represented as follows.

Criterion of pure state. For the functions $W_{[\pm]}(p,q)$ and $W_{\{\pm\}}(p,q)$ to be even and odd components of the Wigner function for charge-invariant observables, it is necessary and sufficient that equalities (29), (32), (33) hold true, and the following conditions are satisfied:

$$\frac{\partial^2}{\partial p_1 \partial p_2} \ln \int\limits_{-\infty}^{+\infty} \varepsilon^{-1}(p', p_1; p'', p_2) W_{[\pm]}(\frac{1}{2}(p'+p''), q) e^{\frac{i}{\hbar}(p'-p'')q} dq dp' dp'' = 0, \tag{34}$$

$$\frac{\partial^2}{\partial p_1 \partial p_2} \ln \int\limits_{-\infty}^{+\infty} \chi^{-1}(p', p_1; p'', p_2) W_{\{\pm\}}(\frac{1}{2}(p'+p''), q) e^{\frac{i}{\hbar}(p'-p'')q} dq dp' dp'' = 0. \tag{35}$$

The analogous criterion in the non-local theory one can obtain if the function $\varepsilon^{-1}(p', p_1; p'', p_2)$ is changed by the product of two δ-functions. However, there exists an example, where the Wigner functions in the standard and non-local theories coincide.

It results directly from the determination of the even part of the Wigner function (23) and from the evident fact of $\varepsilon(n, n) = 1$, that the Wigner function of the stationary states coincides with one in the non-local theory. Therefore, the peculiarities, visualized by this criterion, can manifest themselves in the non-stationary processes only.

Hence, the ε-factor influences the value of interference terms only. Taking into account the fact that $\varepsilon(n, m) > 1$, if $n \neq m$, one can say, that there is an effective increase of the coherence between eigenstates of the Hamiltonian.

V. CONCLUSIONS

In this paper we have briefly considered the peculiarities of the Wigner function formalism, that result from the nontrivial charge structure of the coordinate and momentum operators. It should be noticed, that in this connection, there is another problem in the relativistic Wigner function formalism, related to the Lorentz invariance. We have not considered this topic here. However, this question is very interesting, and it can be related to fundamental concepts of quantum physics [8].

Non-trivial charge structure of the coordinate and momentum operators leads to appearance of additional multipliers (ε-factor) for the interference terms between eigenstates of the Hamiltonian. It means the effective increase of the coherence; i.e. information about the relative phase between coefficients $C_{n;\pm}$ becomes more evident.

To be more correct, we say that this is a peculiarity of the relativistic coordinate (momentum) measurement. The coherence does not increase objectively. Hence, this effect makes better “observed characteristic” of the interference terms only.

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