

# Steady State Entanglement in Cavity QED

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**Abstract:** We investigate steady state entanglement in an open quantum system, specifically a single atom in a driven optical cavity with cavity loss and spontaneous emission. The system reaches a steady pure state when driven very weakly. Under these conditions, there is an optimal value for atom-field coupling to maximize entanglement, as larger coupling favors a loss port due to the cavity enhanced spontaneous emission. We address ways to implement measurements of entanglement witnesses and find that normalized cross-correlation functions are indicators of the entanglement in the system. The magnitude of the equal time intensity-field cross correlation between the transmitted field of the cavity and the fluorescence intensity is proportional to the concurrence for weak driving fields.

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## 1. Introduction

The study of entanglement has emerged as a central theme of quantum physics in recent years. It is driven both by fundamental questions and by the increasing interest in applications that go beyond the limit of classical physics. Entanglement as a measurable quantity is a complicated subject, in particular when the systems have multiple components. Here we choose to study entanglement and its possible avenues of quantification in an open quantum system. This system, the canonical model of cavity QED [1], has a single atom coupled to the mode of an optical cavity with two reservoirs or avenues for extracting information: spontaneous emission and losses from the cavity.

Two particles (or systems),  $A$  and  $B$  are said to be in an entangled state if the wave function of the complete system does not factorize, that is  $|AB\rangle \neq |A\rangle|B\rangle$ . One consequence of this form of the wavefunction is that a measurement on system  $A$  yields information about system  $B$  without any direct interaction with system  $B$ . For systems with the same dimension, in particular, a (pure) state is said to be maximally entangled if tracing over one of the two systems, say  $A$ , leaves the other one in a totally mixed state; this means that one can gain complete knowledge of system  $B$  by performing measurements on  $A$  only. An example that is of relevance to this work is the maximally entangled state of an atom and a field mode,  $|\Psi\rangle = (1/\sqrt{2})(|1, g\rangle + |0, e\rangle)$  with the first index denoting the number of photons in the field mode and the second ( $e = \textit{excited}$ ,  $g = \textit{ground}$ ) denoting the state of the atom. A measurement of the state of the atom immediately tells us the number of photons in the field mode; or a measurement of the photon number immediately tells us the state of the atom.

The von Neumann entropy  $E = -\text{tr}_A(\rho_A \log_2 \rho_A)$  of the reduced density matrix of system  $A$ ,  $\rho_A = \text{tr}_B(\rho_{AB})$  [2] quantifies the amount of entanglement in a given bipartite quantum system

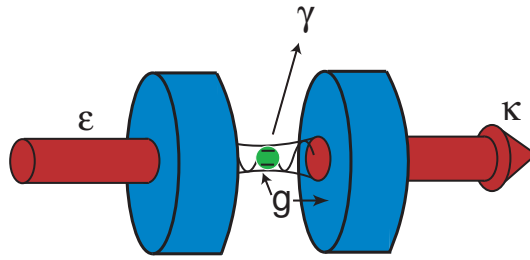


Fig. 1. Single atom in a weakly driven optical cavity. Here  $g$  is the reversible coupling rate between the mode of the cavity and the atom,  $\kappa$  is the decay rate of the field mode of the cavity,  $\gamma$  is the spontaneous emission rate.  $\varepsilon$  is the external drive (taken to be a classical field).

in a pure state. For mixed states, on the other hand, although it is easy enough to define what is meant by a totally unentangled state—namely, one in which it is possible to represent the density operator as an incoherent superposition of factorizable states—quantifying the amount of entanglement in a partially entangled state is not, in general, simple. The natural generalization of the pure-state measure indicated above, known as the entanglement of formation, utilizes a decomposition of the quantum state  $\rho = \sum_j P_j |\psi_j\rangle\langle\psi_j| = \sum_j P_j \rho_j$ , and then defines  $E = \min(\sum_j P_j E_j)$  where  $E_j$  is the von Neumann entropy for the density matrix  $\rho_j = |\psi_j\rangle\langle\psi_j|$ , and the minimum is taken over all the possible decompositions, which is in general a very challenging task [2, 3]. As a result of this, alternative measures have been proposed, such as the logarithmic negativity [4]. It is also possible that some particular measurement scheme may result in a most natural unraveling of the density operator, in the sense of the quantum trajectories approach [5] (especially for systems that are continually monitored), and in that case it may be physically meaningful to focus only on the entanglement of the (conditionally pure) states obtained via that particular unraveling.

One of the main purposes of this paper is to determine how much information about the atom-field entanglement in our canonical cavity QED system can be gleaned from the kinds of measurements represented by the traditional correlation functions of quantum optics. As we shall show below, we are actually able to avoid the difficulties for mixed-state entanglement because, in the limit we are interested in, our system is, to a good approximation, in a pure state, in spite of its being an open system interacting with two reservoirs.

## 2. Cavity QED system

Fig. 1 shows a two level atom in a driven optical cavity. We consider a single-ended cavity, with the intracavity field decaying via the output mirror at rate  $\kappa$ . The two-level atom has a spontaneous emission rate to modes out the sides of the cavity denoted by  $\gamma$ , which is generally less than the free space Einstein  $A$  coefficient. The resonant coupling between the atom and the field mode is given by  $g = \mu_{eg} \sqrt{\omega/2\hbar\varepsilon_0 V}$  with  $\mu_{eg}$  the electric dipole matrix element,  $\omega$  the transition frequency and  $V$  the volume of the cavity mode. The driving field is taken to be a large classical field  $\varepsilon$  incident on the input mirror, with small transmission  $T_{in}$ , so that the incident flux (in photon units) inside the cavity is proportional to  $T_{in}\varepsilon^2$

The quantum trajectory wave function that characterizes the system under a non-Hermitian

Hamiltonian is:

$$|\Psi_c(t)\rangle = \sum_n^{\infty} (C_{g,n}(t)e^{-iE_{g,n}t}|g,n\rangle + C_{e,n}(t)e^{-iE_{e,n}t}|e,n\rangle) \quad (1)$$

$$H = \hbar g (a^\dagger \sigma_- + a \sigma_+) - i\kappa a^\dagger a - i\frac{\gamma}{2} \sigma_+ \sigma_- + i\hbar \varepsilon (a^\dagger - a) \quad (2)$$

with collapse operators

$$\mathcal{A} = \sqrt{\kappa} a \quad (3)$$

$$\mathcal{S} = \sqrt{\frac{\gamma}{2}} \sigma_- \quad (4)$$

associated with photons exiting the output mirror and spontaneous emission out the side of the cavity. The indices  $e(g)$  indicate the atom in the excited (ground) state, while  $n$  is the number of photons in the mode. The energies are  $E_{e,n} = E_{g,n+1} = \hbar\omega(n+1/2)$ . We have the usual creation ( $a^\dagger$ ) and annihilation ( $a$ ) operators for the field, and Pauli raising and lowering operators  $\sigma_\pm$  for the atom.

In the weak driving limit, the system reaches a steady-state wave function:

$$|\Psi\rangle = |0g\rangle + A_{1,g}|1g\rangle + A_{0,e}|0e\rangle + A_{2,g}|2g\rangle + A_{1,e}|1e\rangle \quad (5)$$

where the  $A_{ij}$  are known [6, 7]. They are

$$A_{1,g} = \alpha \quad (6)$$

$$A_{0,e} = \beta \quad (7)$$

$$A_{1,e} = \alpha\beta q \quad (8)$$

$$A_{2,g} = \alpha^2 pq / \sqrt{2}. \quad (9)$$

The quantities  $p$  and  $q$  would be 1 for coupled harmonic oscillators. In cavity QED they differ from unity due to the non-harmonic, or saturable, nature of the atom. The squares of coefficients of single excitation  $A_{1,g}$ ,  $A_{0,e}$  give the rates of detection of single photons through the output mirror or in fluorescence (steady state), while the squares of the double excitation coefficients  $A_{1,e}$ ,  $A_{2,g}$  give the rates of detection of two photons either in coincidence (one through the mirror, and one in fluorescence) or both out of the mirror. The variables are

$$\alpha = \frac{\varepsilon}{\kappa(1+2C_1)} \quad (10)$$

$$\beta = \frac{-2g}{\gamma} \alpha \quad (11)$$

$$p = 1 - 2C'_1 \quad (12)$$

$$q = \frac{(1+2C_1)}{(1+2C_1-2C'_1)} \quad (13)$$

$$C_1 = \frac{g^2}{\kappa\gamma} \quad (14)$$

$$C'_1 = C_1 \frac{2\kappa}{(2\kappa+\gamma)} \quad (15)$$

The one-excitation amplitudes  $A_{1,g}$  and  $A_{0,e}$  are proportional to the driving field  $\varepsilon$ ; the two-excitation amplitudes  $A_{2,g}$ , and  $A_{1,e}$  are proportional to the square of the driving field,  $\varepsilon^2$ . [6].

The norm of this wave function is  $|\Psi\rangle = \sqrt{1 + O(\varepsilon^2)}$ ; hence to lowest order in  $\varepsilon$ , the coefficient of the vacuum should be  $(1 - (1/2)O(\varepsilon^2))$ . The term  $O(\varepsilon^2)$  makes no contribution to lowest nonzero order in  $\varepsilon$  for the correlation functions or entanglement measures considered here.

The entanglement of formation for this system is calculated from the density matrix after tracing over the field variables:

$$\rho_{atom} = Tr_{field} |\Psi\rangle\langle\Psi| \quad (16)$$

$$= \begin{pmatrix} 1 + A_{1,g}^2 + A_{2,g}^2 & A_{1,e}A_{1,g} + A_{0,e} \\ A_{1,e}A_{1,g} + A_{0,e} & A_{1,e}^2 + A_{0,e}^2 \end{pmatrix} \quad (17)$$

The eigenvalues of this matrix are, to lowest nonvanishing order,

$$\begin{aligned} \lambda_1 &= (A_{1,g}A_{0,e} - A_{1,e})^2 \\ &= |A_{1,g}|^2 |A_{0,e}|^2 (q - 1)^2 \\ &= \left(\frac{\varepsilon}{\kappa}\right)^4 \xi^2 \end{aligned} \quad (18)$$

$$\begin{aligned} \lambda_2 &= 1 - (A_{1,g}A_{0,e} - A_{1,e})^2 \\ &= 1 - \left(\frac{\varepsilon}{\kappa}\right)^4 \xi^2 \end{aligned} \quad (19)$$

$$(20)$$

where  $q$  is defined in Eq. (13), and we have defined

$$\xi = \frac{2g}{\gamma(1 + 2C_1)^2} (q - 1) \quad (21)$$

The entropy  $E = -\lambda_1 \log_2 \lambda_1 - \lambda_2 \log_2 \lambda_2$  is then (again to lowest leading order)

$$\begin{aligned} E &= -\left(\frac{\varepsilon}{\kappa}\right)^4 \xi^2 \log_2 \left[ \left(\frac{\varepsilon}{\kappa}\right)^4 \xi^2 \right] - \left(1 - \left(\frac{\varepsilon}{\kappa}\right)^4 \xi^2\right) \log_2 \left[ 1 - \left(\frac{\varepsilon}{\kappa}\right)^4 \xi^2 \right] \\ &\approx -\left(\frac{\varepsilon}{\kappa}\right)^4 \xi^2 \left( \log_2 \left[ \left(\frac{\varepsilon}{\kappa}\right)^4 \right] + \log_2 [\xi^2] - 1 \right) \\ &\approx -\left(\frac{\varepsilon}{\kappa}\right)^4 \log_2 \left[ \left(\frac{\varepsilon}{\kappa}\right)^4 \right] \xi^2. \end{aligned} \quad (22)$$

where we have taken the weak field limit,  $\varepsilon$  being the smallest rate in the problem, so  $\varepsilon/\kappa \ll 1$ . The approximation (22) will hold provided  $(\varepsilon/\kappa)^2 \ll |\xi|$ .

This entropy is the same as that obtained by using the density matrix for the field alone, traced over the atomic degrees of freedom.

The concurrence, first introduced by Wootters for two qubits[3], can also be used to characterize entanglement between two quantum systems of arbitrary dimension [8, 9, 10, 11]. The concurrence for our system is

$$\begin{aligned} \mathcal{C} &= \sqrt{2(1 - Tr \rho_{atom}^2)} \\ &= \sqrt{4(A_{1,g}A_{0,e} - A_{1,e})^2} \\ &= 2 \left(\frac{\varepsilon}{\kappa}\right)^2 |\xi| \end{aligned} \quad (23)$$

To see why  $|\xi| \propto |A_{1,e} - A_{0,e}A_{1,g}|$  may be a good indication of entanglement, consider what happens if the wavefunction is a product state. We could write

$$\begin{aligned} |\Psi\rangle_P &= |\psi_F\rangle \otimes |\phi_A\rangle \\ &= (D_0|0\rangle + D_1|1\rangle + D_2|2\rangle) \otimes (C_g|g\rangle + C_e|e\rangle) \\ &= D_0C_g|0g\rangle + D_1C_g|1g\rangle + D_0C_e|0e\rangle + D_2C_g|2g\rangle + D_1C_e|1e\rangle \end{aligned} \quad (24)$$

For weak excitations, the coefficient of the ground state of the system is  $D_0C_g = 1$ , or  $C_g = D_0 = 1$ . Then the product state is

$$|\Psi\rangle_P = |0g\rangle + D_1|1g\rangle + C_e|0e\rangle + D_2|2g\rangle + D_1C_e|1e\rangle \quad (25)$$

Just knowing the one excitation amplitudes does not yield any information about entanglement, as it is possible to have  $A_{1,g} = D_1$  and  $A_{0,e} = C_e$ .  $A_{2,g}$  gives no information about entanglement, just nonclassical effects in the field, as it only involves field excitation. For weak fields  $D_2$  is exactly  $A_{2,g}$ . The entanglement shows up in the value of  $A_{1,e}$ ; if this value does not satisfy  $A_{1,e} = D_1C_e = A_{0,e}A_{1,g}$ , then it is not possible to write the state as a product state.

In the presence of a non-zero vacuum contribution (as any real quantum state will have), one can learn nothing about entanglement simply by measurement of one-excitation amplitudes or probabilities. For example, the state  $|0, g\rangle + \alpha(|1, g\rangle + |0, e\rangle)$  is entangled, but only if one is certain that the probability amplitudes for higher excitation are truly zero. A state of the form  $|0, g\rangle + \alpha(|1, g\rangle + |0, e\rangle) + O(\epsilon^2)$  cannot be said to be entangled without information on the relative size of the probability amplitude  $A_{1,e}$ . Measurement of one-excitation amplitudes conditioned by a previous measurement *can* yield information about entanglement. This can be accomplished by utilizing cross-correlation functions. A first important conclusion out of this study is that a measure of the zero time cross correlation between the atom and the field, as well as the mean transmitted and fluorescent intensities yields a measure of entanglement in the weak field limit.

### 3. Entanglement for weak excitation

Equation (22) of the previous section gives the amount of entanglement in the system as a function of the one and two excitation amplitudes. In terms of specific system parameters the concurrence is:

$$\mathcal{C} = |2\alpha\beta(q-1)| = \frac{16g^3\epsilon^2\kappa}{(2g^2 + \gamma\kappa)^2(2g^2 + \kappa(\gamma + 2\kappa))}. \quad (26)$$

This section analyzes the sensitivity of the concurrence to the different parameters that appear in Eq. (26), while trying to give physical reasons for their influence on the entanglement. Despite the fact that the rates of decay could be the same through the two reservoirs, spontaneous emission ( $\gamma$ ) reduces entanglement more than cavity loss ( $\kappa$ ). This is due to the fact that a  $\gamma$  event (spontaneous emission) *must* come from the atom, while a  $\kappa$  event (cavity transmission) could come from either the drive or a photon emitted by the atoms into the cavity mode. A spontaneous emission event unambiguously leaves the atom in the ground state, and the system wavefunction factorizes.

Fig. 2 shows a remarkable result in the entanglement of the system as a function of the three rates in the problem. There is an optimal value for the coupling constant  $g$  given a set of dissipation rates  $\kappa, \gamma$ . For many interesting cavity QED effects, stronger coupling is generally better, such as the enhancement of the spontaneous emission by a factor of  $1 + 2C_1 = 1 + 2g^2/\kappa\gamma$  (this formula strictly holds only in the bad cavity limit  $\kappa \gg g, \gamma$ ). However, here increasing

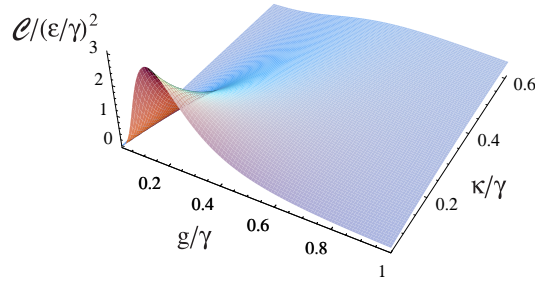


Fig. 2. A plot of  $\mathcal{C}$  scaled by  $(\varepsilon/\gamma)^2$  as a function of  $\kappa/\gamma$  and  $g/\gamma$  for weak excitation.

the coupling of the atom and field mode eventually decreases the amount of entanglement. To explain this it is instructive to recall that the concurrence  $\mathcal{C} = |2\alpha\beta(q-1)|$ , where  $\alpha$  is the mean cavity field, and  $\beta = -g\alpha/\gamma$  is the mean atomic dipole. As the coupling  $g$  increases, for a fixed weak driving field  $\varepsilon$ , the intracavity field  $\alpha = \varepsilon/(\kappa + 2g^2/\gamma)$  decreases. The intracavity field is the sum of the driving field in the cavity  $\varepsilon/\kappa$ , and the field radiated by the atom,  $(-2C_1/(1+2C_1))\varepsilon/\kappa$ , the minus sign resulting from the fact that the radiated field is  $\pi$  out of phase with the driving field on resonance. We see that as  $g$  and  $C_1$  increase, the intracavity field decreases. This means that the steady-state wavefunction has a larger vacuum component, and consequently less entanglement. Another way to view this is that the cavity enhancement of the spontaneous emission rate means a larger loss rate for the system as the coupling increases, which is bad for entanglement.

More formally, consider what happens if the two-excitation amplitudes in Eq. (5) are arbitrarily set to zero, which amounts to setting  $q = 0$  in Eq. (26), in which case the entanglement is only determined by the prefactor  $|\alpha\beta|$ . The steady-state wave function becomes

$$|\psi\rangle_{ss} = |0g\rangle + \alpha(|1g\rangle - \frac{g}{\gamma}|0e\rangle). \quad (27)$$

There are two interesting limits on this Eq. (27) for the parameter  $f = g/\gamma$ . If  $f \gg 1$ , the steady state wavefunction is approximately  $|\psi\rangle_{ss} = |0\rangle(|g\rangle - f\alpha|e\rangle)$  which is a product state. Also, if  $f \ll 1$ , the steady state wavefunction is approximately  $|\psi\rangle_{ss} = |g\rangle(|0\rangle + \alpha|1\rangle)$  which again is a product state. To have entanglement between the atom and cavity mode, we must have the parameter  $f \simeq 1$ , so as to prepare a steady state wavefunction of the form  $|\psi\rangle_{ss} = |0g\rangle + \alpha(|1g\rangle - |0e\rangle) = |0g\rangle + \alpha|-\rangle$ , a mixture of the vacuum with a small entangled state component.

The decrease of the prefactor  $|\alpha\beta|$  is the dominant reason why the concurrence decreases with increasing  $g$  for large coupling. Close inspection of Fig. 2 also shows that there is an optimal cavity loss rate  $\kappa$  for entanglement for a fixed  $g$  and  $\gamma$ . This is a result of reaching a maximum in the population of the states different from the vacuum (Eq. (5)). Our results here are consistent with the numerical results of Nha and Carmichael [5].

When the system is driven off resonance, its response is typically characterized by transmission and fluorescent spectra [12, 13]. Although these are important probes of the system, they do not, in this limit, carry information about the entanglement, since they are derived from only the one-excitation amplitudes.



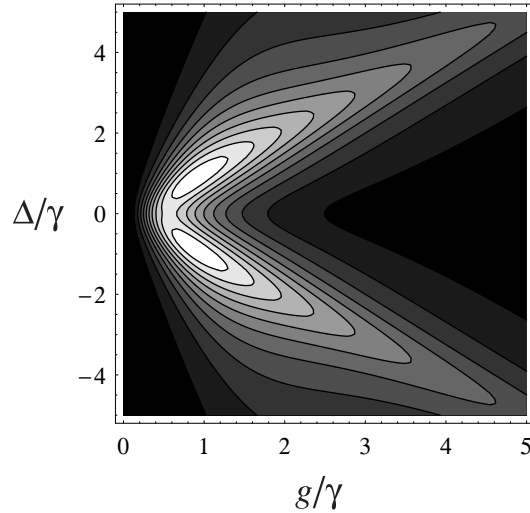


Fig. 3. Contour plot of  $\mathcal{C}$  as a function of  $g/\gamma$  and  $\Delta/\gamma$  for  $\kappa/\gamma = 0.5$

The concurrence as a function of the detuning of the driving laser shows that the steady state entanglement decreases typically by a factor of  $1/\Delta^3$  for large detuning, where  $\Delta = (\omega - \omega_l)$  with  $\omega$  the resonant frequency of the atom and cavity, and  $\omega_l$  the frequency of the driving probe laser. But in the case where  $g$  is larger than  $\kappa$  and  $\gamma$ , the response is maximized at the vacuum-Rabi peaks [14]. Figure 3 shows a contour plot of  $\mathcal{C}$  for parameters in the regime of cavity QED where the two decay rates are similar:  $2\kappa/\gamma = 1.0$ . The concurrence increases with increasing  $g$  on resonance up to a saddle point, and then decreases. However the entanglement persists for detunings on the order of  $g$ , the approximate location of the vacuum-Rabi peaks in the spectra of the system.

Detuning to a vacuum-Rabi peak ( $\Delta = \pm g$ ), generates a steady state wave function of the form

$$|\psi\rangle_{ss} = |0, g\rangle + \alpha\Gamma_1(g/\gamma)|1, \pm\rangle + \alpha^2\Gamma_2(g/\gamma)|2, \pm\rangle, \quad (28)$$

where  $|n, \pm\rangle = (1/\sqrt{2})(|n, g\rangle \pm |n-1, e\rangle)$  is the  $n$  photon dressed atom-field state one is tuned near and  $\Gamma_1(g/\gamma)$  and  $\Gamma_2(g/\gamma)$  are functions that are maximal when  $g \simeq \gamma$ . This is a state of mainly vacuum, plus a part that has entanglement between the atom and the cavity. It would seem that by continuing to tune to a vacuum-Rabi peak as  $g$  increases, it would be possible to maintain the entanglement, but Fig. 3 shows that this is not the case. Rather, as argued (for the on-resonance case) above, the crucial parameter for maximizing entanglement is  $f = g/\gamma \propto 1/\sqrt{n_{sat}}$ , where  $n_{sat} = \gamma^2/8g^2$  is the saturation photon number. This is the dependence on the nonlinearity of the atomic system. Recall that, if these were two driven coupled harmonic oscillators,  $q = 1$  and there would be no entanglement. A nonlinear interaction between the two harmonic oscillators would be needed to entangle them, as in the signal and idler modes in optical parametric oscillation. This nonlinear interaction would generate two-mode squeezing, which could be measured by homodyne detection of mode A(B) conditioned on detection of a photon in mode B(A), just as squeezing in one mode can be detected via conditioned homodyne detection of a mode based on a photodetection from that mode[15, 16]. The nonlinearity of the two-level atom is needed to generate two-mode squeezing and entanglement between the atom and the cavity field. Even though the driving field is weak and the atom never nears saturation, there can only be entanglement with a linear atom-field coupling if the atom has a nonlinear



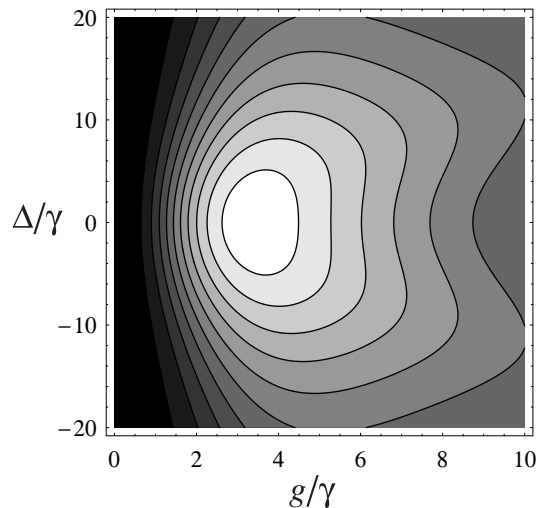


Fig. 4. Contour plot of  $\mathcal{C}$  as a function of  $g/\gamma$  and  $\Delta/\gamma$  for  $\kappa/\gamma = 10$

response, as two-level atoms do.

The concurrence shows its sensitivity to different parameters. Fig. 4 shows a contour plot of  $\mathcal{C}$  versus  $g/\gamma$  and  $\Delta/\gamma$  for a case where the cavity decay rate is larger than the spontaneous emission rate ( $\kappa/\gamma = 10.0$ ). The entanglement is largest near  $g/\gamma = 4.0$ , before the vacuum-Rabi splitting of the spectrum, which does not occur in this case until  $g/\gamma \sim 10.0$ , at which point the entanglement is already diminishing. The size of the maximum concurrence decreases by increasing  $\kappa/\gamma$  from 0.5 to 10.0 by a factor of about 30.

#### 4. Measurements of entanglement with correlation functions

The calculation of entanglement leads now to the question of how to implement measurements that give the full information in the case of this cavity QED system under weak excitation. The previous section shows that the concurrence is related to the rate of single photon counts out of the cavity or in fluorescence and to the rate of coincident counts from the cavity and fluorescence. These are the quantities associated in quantum optics with correlation functions, first introduced by Glauber [17, 18, 19, 20]. Generally these correlation functions involve comparing a field (intensity) of one mode with the field (intensity) of the same mode at a later time (or different spatial location), with some exceptions [21, 22, 23, 24, 25, 26]. However, entanglement in cavity QED has two components: atom and cavity mode. It is natural to look at cross correlations between the cavity mode and the fluorescent light that falls in the mode of the detector.

Consider a general cross-correlation function for two-modes of the electromagnetic field:

$$G = \langle f_1(b^\dagger, b) f_2(a^\dagger, a) \rangle / \langle f_1(b^\dagger, b) \rangle \langle f_2(a^\dagger, a) \rangle. \quad (29)$$

with  $f_1$  and  $f_2$  well behaved functions, in the sense of a convergent Taylor series on the Hilbert space of interest. If  $|\psi\rangle$  is a product state, the correlation function  $G(a, b)$  factorizes and then is unity. If it is *not* a product state, then this will manifest itself in a non-unit value for the normalized cross-correlation functions.

The simplest cross correlation function to consider is  $g_{TF}^{(1)}(0)$ . This could be obtained by

measuring the visibility of the fringe pattern formed by interfering the transmitted and fluorescent light. For the weakly driven cavity-QED system, this is

$$\begin{aligned}
 g_{TF}^{(1)}(0) &= \frac{\langle \sigma_+ a \rangle}{\langle \sigma_+ \rangle \langle a \rangle} \\
 &= \frac{\alpha \beta}{\alpha \beta} \\
 &= 1
 \end{aligned} \tag{30}$$

so to lowest order, there is no information in this correlation function about entanglement.

To obtain information about entanglement the correlation function has to probe the two-excitation part of the state. A possibility to do this is the intensity cross correlation:

$$\begin{aligned}
 g_{TF}^{(2)}(0) &= \frac{\langle \sigma_+ a^\dagger a \sigma_- \rangle}{\langle a^\dagger a \rangle \langle \sigma_+ \sigma_- \rangle} \\
 &= \frac{|A_{1e}|^2}{|A_{1g} A_{0e}|^2} \\
 &= q^2
 \end{aligned} \tag{31}$$

This normalized correlation function is directly related to the coefficient of double excitations (See Eqs. (5), (8), (13)). If  $q = 1$  then  $g_{TF}^{(2)}(0) = 1$  and there is no entanglement; so a non-unit value of  $q$  indicates entanglement. Using second-order intensity correlations has been proposed in the context of entangled coherent states by Stobińska and Wódkiewicz [27].

The cross correlation function  $g_{TF}^{(2)}(0)$  contains information about the average photon number *in coincidence* with a measurement of the fluorescence relative to the average photon number in the absence of any interrogation of the fluorescence.  $g_{TF}^{(2)}(0) - 1 = q^2 - 1$  is an indicator of entanglement.

A way to measure  $q$  directly utilizes a field-intensity correlation function  $h_\theta(\tau)$  [28], that can be implemented as a homodyne measurement conditioned on the detection of a fluorescent photon,

$$\begin{aligned}
 h_{\theta=0}^{TF}(0) &= \frac{\langle I_F E_T \rangle}{\langle I_F \rangle \langle E_T \rangle} \\
 &= \frac{\langle (a^\dagger + a) \sigma_+ \sigma_- \rangle}{\langle a^\dagger + a \rangle \langle \sigma_+ \sigma_- \rangle} \\
 &= \frac{A_{1,e}}{A_{0,e} A_{1,g}} \\
 &= q
 \end{aligned} \tag{32}$$

So  $h_{\theta=0}^{TF}(0) - 1 = q - 1$  is also an indicator of entanglement in this system. What makes this measurement possible experimentally is the conditioning that selects only times when there is a fluctuation and the rest of the time (when the vacuum is present) no data is collected [16]. For one mode, the homodyned transmitted field conditioned by detection of a photon from that mode, is a measure of squeezing in that mode [28]. A homodyne measurement of the transmitted field conditioned by detection of a fluorescent photon is a measure of the two-mode squeezing, with the cavity field and atomic dipole as the two components. Generally, two-mode squeezing is an indicator of entanglement between the two modes. Gea Banacloche *et al.* explored this correlation function in a different regime of cavity QED and found it to be a witness of the dynamics of entanglement [29].

Non-classicality and entanglement are not necessarily simultaneously present. For example for two oscillators one could have  $|\psi\rangle = (1/\sqrt{2})(|A,B\rangle + |B,A\rangle)$ , where  $A$  and  $B$  are coherent state amplitudes. In this state, there is entanglement, but each individual mode shows no non-classical behavior. Conversely, one can have non-classical behavior with no entanglement, say for example the atom in the ground state and the field in a squeezed coherent state.

There is a particular form of the Schwartz inequality that must be satisfied for a classical field for the specific case of the system we are considering here:

$$(g_{TF}^{(2)}(0) - 1)^2 \leq |(g_{TT}^{(2)}(0) - 1)(g_{FF}^{(2)}(0) - 1)|, \quad (33)$$

Here  $TT$  and  $FF$  denote zero delay intensity correlations for the transmitted and fluorescent fields respectively. In the one-atom limit,  $g_{FF}^{(2)}(0) = 0$ , and  $g_{TT}^{(2)}(0) = q^2 p^2$ , so this inequality becomes  $|q^2 - 1|^2 \leq |q^2 p^2 - 1|$  which depends on  $q$ , but also on the parameter  $p$  (Eq. (12)), which can be varied independently. There is no one-to-one relationship between Schwarz inequality violations and entanglement (by this measure) in this particular system.

## 5. Conclusion

We find that entanglement in weakly driven cavity QED is characterized by comparison of two-excitation probability amplitudes to single excitation amplitudes, in particular the amplitude involving one excitation in each subsystem. It is necessary to have a small saturation photon number to enhance the nonlinear response which generates a larger entanglement. But this is true only to a point. We find the maximal entanglement for small  $\kappa$  and when  $g/\gamma$  is on the order of unity. This stems from the dual role of the coupling  $g$ . It couples energy into the atom, but due to cavity enhanced spontaneous emission, it can also channel energy out.

Increasing  $\gamma$  decreases the entanglement, and this can be explained in terms of the effect of the two decay processes on the system. If we detect a fluorescent photon we know it has come from the atom, and the atom is in the ground state. If we obtain a transmitted photon, it could have been emitted from the atom into the cavity mode, or just be a driving field photon that has passed through the cavity without interaction with the atom. It is the interference of these two indistinguishable processes that leads to nonclassical effects in the transmitted field.

We have found a variety of cross-correlation functions that are indicators, or witnesses, of entanglement in this system. One can learn nothing about the entanglement by examining only one- or two- excitation amplitudes separately. In particular we find that a measurement of two-mode squeezing, or a homodyne measurement of the transmitted field conditioned on the detection of a fluorescence photon is directly proportional to the entanglement calculated via the reduced von Neumann entropy. Further work remains to generalize this approach to situations with higher drives, but the general approach of looking at entanglement together with the specific correlation function to measure gives physical insight into this problem.

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