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Field fluctuations measured by interferometry*

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Abstract
We derive the complete photon count statistics of an interferometer based on two beam splitters. As a special case we consider a joint intensity–electric field measurement. Our approach is based on the transformation properties of state vectors as well as field operators at a beam splitter.

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1. Introduction

The dream of a quantum internet [1] that is a network where 'flying' qubits transmit quantum information [2] between several nodes processing this information has come a long way[7]. Indeed, by now several elements of such an arrangement have been realized experimentally (see for example [4]) using ions in traps, entangled photons or a cavity QED. Despite this breathtaking progress in quantum technology, there is still a lot to be learned and we recall the warning of Thomas Henry Huxley (1825–1895) who wrote in 1868 in an inaugural address titled *'A Liberal Education; and Where to Find it'* [5]:

> 'The chess board is the world, the pieces are the phenomena of the universe, the rules of the game are what we call the laws of nature. The player on the other side is hidden from us. We know that his play is always fair, just, and patient. But also we know, to our cost, that he never overlooks a mistake, or makes the smallest allowance for ignorance. To the man who plays well the highest stakes are paid with that sort of overflowing generosity with which the strong shows delight in strength. And one who plays ill is checkmated—without haste, but without remorse.'

With this warning in mind we concentrate in this paper on a six-port interferometer, which represents an elementary version of a quantum network. Such an analytical device is made of two beam splitters as shown in the rectangle on the left-hand side of figure 1.

The present analysis is in the tradition of earlier work[8] on multi-port interferometers. In the context of an operational approach [7] towards the old question of the phase operator (see for example [8]) in quantum mechanics, the eight-port interferometer operated as a double homodyne detector has been studied extensively [9]. It was shown that in the limit of a strong local oscillator the count statistics of this device is given by the Q-function of the input field. The underlying operators are identical to the ones of the Einstein–Podolsky–Rosen situation [10–12].

* The work presented here was stimulated by discussions during the Lake Garda Conference 2001. The recent experimental interest in six-port interferometry has moved us to return to the problem. We feel, moreover, that the topic is appropriate for the Festschrift in honour of Stig Stenholm since he can truly be considered a pioneer in the field of quantum networks. We hope that our discussion may pique his interest.

[6] Herbert Walther was an active participant in this project in his familiar style. He was also a close friend of Stig's. Herbert remains a co-author although he did not live to see the completion of this manuscript.

[7] For landmark papers on the quantum internet see Cirac *et al* [3] and Kimble [3]; for a recent proposal see Vasilev *et al* [3].

[8] For expositions of the theory of multiports, see for example [6].

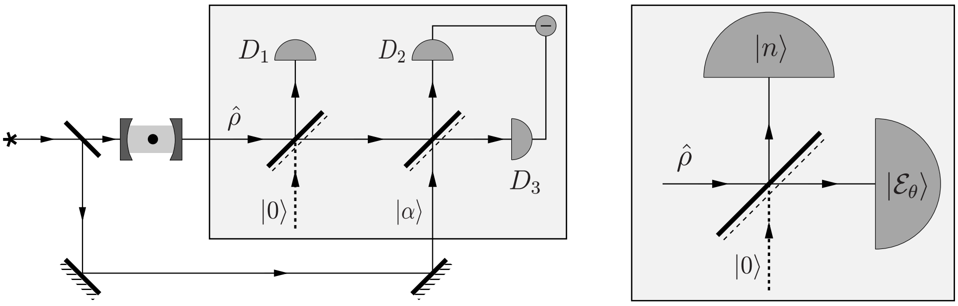

Figure 1. The six-port interferometer shown in the left box analyses the light created, for example, by the interaction of an atom with a driven cavity field. The three inputs are: (i) the field emerging from the cavity described by the density operator $\hat{\rho}$, (ii) the vacuum field denoted by the state $|0\rangle$ and (iii) a local oscillator in a coherent state $|\alpha\rangle$. Detectors D_1, D_2 and D_3 count the excitations of the three output modes. The counts of detectors D_2 and D_3 are subtracted from each other, providing a homodyne measurement. In the limit of a strong local oscillator this setup is equivalent to the one shown in the right box consisting of a single beam splitter with projections of the two output modes on the photon number state $|n\rangle$ and the electric field state $|\mathcal{E}_\theta\rangle$.

1.1. Six-port interferometer setups and experimental applications

In its most elementary version, a single beam splitter consists of two input and two output ports. Thus for two such beam splitters we have a total of four input and four output modes. When we feed one of the output modes of one of the beam splitters into one of the input ports of the other, we arrive at a total of three input and three output ports. This arrangement of two beam splitters constitutes the six-port interferometer discussed in this paper[9].

During the last decades a special version of this device has become rather popular. Here, a joint photon number and an electric field measurement is made. In this situation the three input ports are occupied by: the field to be analysed, the vacuum field and a local oscillator field. On all three output modes intensity measurements are made. However, the photon counts from two of the three detectors are subtracted from each other as indicated in the left box of figure 1.

Setups of this type have been used in many experimental situations. In the early days of squeezing it was proposed [14] to measure the quadrature distribution of one of the two photons created by a parametric oscillator using the second photon as a trigger. At first sight this arrangement seems to be different from the one discussed in this paper. Indeed, it involves only a single beam splitter to make the quadrature measurement by mixing the field to be measured with the local oscillator. However, the missing beam splitter is represented by the nonlinear medium.

Experiments [15, 16] following this recipe [14] have reconstructed the Wigner function of a single Fock state with the help of the Radon transform [17] of the measured quadrature field distributions. Moreover, this concept of a heralding photon underlying the six-port interferometer has also been applied successfully to induce conditional coherence and conditional-phase switch [18] as well as the generation of ultrafast single photons in pure quantum states [19]. Following earlier theoretical ideas [20] recent experiments have created Schrödinger kittens [21] as well as non-local superpositions of quasi-classical light states [22] using conditional measurements of photon number and electric field using a six-port interferometer.

A different class of experiments uses six-port interferometers to study the wave–particle nature of the radiation created in a cavity QED experiment [23–27] shown on the left of figure 1. An atom in a high-Q resonator driven by an external laser field interacts (see for example [28]) with a single mode of the cavity field. The so-created electromagnetic field is coupled out of the resonator and fed into a six-port interferometer operated in the homodyne mode. Recently, a similar experiment [29] in the weak local oscillator limit has been performed analysing the radiation from a single ion stored in a trap. Moreover, a related setup has been used to bridge particle and wave sensitivity in a configurable detector [30].

So far these experiments have been discussed in the framework of the quantum Monte Carlo method [23, 25] and weak measurements [31]. These techniques also yield in a natural way the density operator of the field created by the interaction of the atom with the light in the cavity [27]. Since in the present paper we are interested in the measurement rather than the generation aspects, it is convenient to pursue a more direct evaluation of the count statistics starting from the beam splitter transformations (see for example [32]) occurring in the six-port interferometer.

1.2. Outline of the paper

Our paper is organized as follows: in section 2, we provide a description of the six-port interferometer in terms of state vectors. This method yields immediately the complete photon count statistics of this device. We then turn in section 3 to a six-port interferometer in homodyne mode. In the limit of a strong local oscillator this arrangement is equivalent to a single beam splitter provided that we make a photon number and an electric field measurement on the two output modes. Moreover, we derive an expression for the average count rate.

In contrast to the state vector technique pursued in the first part of the paper, the description of the six-port interferometer presented in section 4 is in terms of correlation functions of electric field operators (see for example [33]). This approach is valid for arbitrary delay times between the two measurements. In particular, we show that due to the

[9] Other versions of six-port interferometers have been suggested. For example, in [13] it was shown that a six-port interferometer consisting of three beam splitters can be used to reconstruct the full quantum state of one of the three input fields.

coherent state of the local oscillator, the correlation function $g^{(2)}$ reduces to one that can be called $g^{(1.5)}$.

The more familiar notation $g^{(n)}$ for the correlation function dates from an era in which all measurements were measurements of fluctuating intensities of optical fields. The integer n reflects the fact that we deal with the correlation between n creation operators with n annihilation operators. The experimental arrangement of the six-port interferometer, as we shall see, allows the measurement of the correlations between two creation operators and one annihilation operator (and vice versa). We resolve this dilemma of notation by calling such correlation functions $g^{(3/2)} = g^{(1.5)}$.

We conclude in section 5 with a brief summary of our results.

2. Complete photon count statistics

In this section, we provide the foundation for our analysis of the six-port interferometer operated in a homodyne mode discussed in the next section. We derive the count statistics of the three detectors at the exit modes of the interferometer shown in figure 1. Since coherent states transform at beam splitters like classical fields [32], we use a diagonal representation of a density operator in terms of coherent states, that is, a P-representation, to find the density operator $\hat{\rho}_{\rm out}$ of the three output modes.

The density operator

$$\hat{\rho}_{\rm in} \equiv \int {\rm d}^2\beta\, P(\beta)\, |\beta\rangle\langle\beta| \otimes |0\rangle\langle 0| \otimes |\alpha\rangle\langle\alpha| \qquad (1)$$

of the three input modes consists of the state

$$\hat{\rho} \equiv \int {\rm d}^2\beta\, P(\beta)\, |\beta\rangle\langle\beta| \qquad (2)$$

to be investigated and described by a P-distribution, the vacuum $|0\rangle$ and a coherent state $|\alpha\rangle$ associated with the local oscillator. Here $|\beta\rangle$ denotes coherent states used in the phase space integral of the density operator $\hat{\rho}$.

Since coherent states transform at beam splitters like classical fields, the input density operator $\hat{\rho}_{\rm in}$ given by equation (1) transforms into

$$\begin{aligned}\hat{\rho}_{\rm out} = \int {\rm d}^2\beta\, P(\beta)\, &\left|-\frac{\beta}{\sqrt{2}}\right\rangle_1 \left\langle -\frac{\beta}{\sqrt{2}}\right| \\ &\otimes \left|-\frac{\beta}{2}+\frac{\alpha}{\sqrt{2}}\right\rangle_2 \left\langle -\frac{\beta}{2}+\frac{\alpha}{\sqrt{2}}\right| \\ &\otimes \left|\frac{\beta}{2}+\frac{\alpha}{\sqrt{2}}\right\rangle_3 \left\langle \frac{\beta}{2}+\frac{\alpha}{\sqrt{2}}\right|. \end{aligned} \qquad (3)$$

Here, we have assumed for the sake of simplicity 50:50 beam splitters and have introduced a phase shift of π due to the reflection from a thicker medium.

The expression equation (3) for $\hat{\rho}_{\rm out}$ indicates that the field states in the three exit modes are entangled unless the P-distribution is a delta function, that is, the field is in a coherent state. Therefore, the photon count statistics

$$W(n_1, n_2, n_3) \equiv \langle n_1, n_2, n_3|\hat{\rho}_{\rm out}|n_1, n_2, n_3\rangle \qquad (4)$$

at the three detectors must be correlated. Indeed, when we substitute the density operator $\hat{\rho}_{\rm out}$ given by equation (3) into (4),we arrive at

$$\begin{aligned} W(n_1, n_2, n_3) = \int {\rm d}^2\beta\, P(\beta)\, W_{n_1}\left(-\frac{\beta}{\sqrt{2}}\right) \\ \times W_{n_2}\left(-\frac{\beta}{2}+\frac{\alpha}{\sqrt{2}}\right) W_{n_3}\left(\frac{\beta}{2}+\frac{\alpha}{\sqrt{2}}\right), \end{aligned} \qquad (5)$$

where

$$W_m(\gamma) \equiv |\langle m|\gamma\rangle|^2 = \frac{|\gamma|^{2m}}{m!}\, {\rm e}^{-|\gamma|^2} \qquad (6)$$

denotes the Poissonian photon statistics of a coherent state of amplitude γ.

Hence, the complete count statistics of the six-port interferometer determined by probability $W(n_1, n_2, n_3)$ to find the number of photons n_1, n_2 and n_3 in the three exit modes follows from a phase-space integral. Its integrand consists of the product of the P-distribution of the state to be analysed and the three Poissonian distributions W_{n_1}, W_{n_2} and W_{n_3} corresponding to the counts in the output ports. Due to integration over the coherent state $|\beta\rangle$, it is more convenient to interpret Poissonian W_m as the Husimi Q-function of a photon number state $|m\rangle$ recalling the identity

$$Q_m(\gamma) = \frac{1}{\pi}\, W_m(\gamma)\,. \qquad (7)$$

In this sense the count statistics

$$\begin{aligned} W(n_1, n_2, n_3) = \pi^3 \int {\rm d}^2\beta\, P(\beta)\, Q_{n_1}\left(-\frac{\beta}{\sqrt{2}}\right) \\ \times Q_{n_2}\left(-\frac{\beta}{2}+\frac{\alpha}{\sqrt{2}}\right) Q_{n_3}\left(\frac{\beta}{2}+\frac{\alpha}{\sqrt{2}}\right) \end{aligned} \qquad (8)$$

is given by the phase-space integral of the product of the P-distribution and three Husimi Q-functions corresponding to the counts n_1, n_2 and n_3.

3. Joint intensity–electric field measurement

In the preceding section, we have derived an expression for the count statistics of a six-port interferometer when we retain the complete information about the photon counts in all exit modes. However, when the interferometer is operated in the homodyne mode, the counts at the detectors D_2 and D_3 are subtracted and the information about the individual counts is erased. In the present section, we first derive an exact expression for the count statistics of a six-port interferometer in the homodyne mode and then consider the strong local oscillator limit. In this case, the interferometer performs a joint intensity–electric field measurement.

3.1. Local oscillator of arbitrary strength

The photon statistics of this device then corresponds to the probability

$$W(n_1, n_{32}) \equiv \sum_{n_2} W(n_1, n_2, n_{32}+n_2) \qquad (9)$$

of finding n_1 quanta of excitation in mode 1 and the corresponding difference $n_{32} \equiv n_3 - n_2$ in modes 2 and 3. Here we sum over the unobserved photon numbers n_2.

When we substitute the explicit expression equation (5) for the photon statistics into equation (9), we find

$$W(n_1, n_{32}) = \int d^2\beta\; P(\beta)\; W_{n_1}\left(-\frac{\beta}{\sqrt{2}}\right) \mathcal{K}_{n_{32}}\left(\frac{\beta}{\sqrt{2}}, \alpha\right), \tag{10}$$

where the homodyne kernel

$$\mathcal{K}_{n_{32}}(\gamma, \tilde{\gamma}) \equiv \sum_{n_2} W_{n_2}\left(\frac{\tilde{\gamma} - \gamma}{\sqrt{2}}\right) W_{n_{32}+n_2}\left(\frac{\tilde{\gamma} + \gamma}{\sqrt{2}}\right), \tag{11}$$

consisting of the sum over the product of two Poisson distributions, equation (6), takes the form [32, 34]

$$\mathcal{K}_{n_{32}}(\gamma, \tilde{\gamma}) = \left|\frac{\tilde{\gamma} + \gamma}{\tilde{\gamma} - \gamma}\right|^{n_{32}} I_{|n_{32}|}\left(|\tilde{\gamma}^2 - \gamma^2|\right) e^{-|\gamma|^2 - |\tilde{\gamma}|^2}. \tag{12}$$

Here I_m denotes the modified Bessel function of mth order.

When we recall the connection equation (7) between the Q-function of a number state and the photon number distribution of a coherent state, the photon statistics of a six-port interferometer in homodyne mode takes the form

$$W(n_1, n_{32}) = \pi \int d^2\beta\; P(\beta)\; Q_{n_1}\left(-\frac{\beta}{\sqrt{2}}\right) \mathcal{K}_{n_{32}}\left(\frac{\beta}{\sqrt{2}}, \alpha\right). \tag{13}$$

It is given by the phase-space integral of the product of the P-distribution, a Husimi Q-function and the homodyne kernel corresponding to the photon number state n_1 and the count difference n_{32}, respectively.

3.2. Strong local oscillator limit

We emphasize that the expression equation (10) for $W(n_2, n_{32})$ is valid for an arbitrary amplitude $|\alpha|$ of the coherent field. When $|\alpha|$ is large, that is, $1 \ll |\alpha|$, a reference phase is established. We now consider the count statistics equation (10) in this limit.

The asymptotic expansion

$$I_n(x) \approx \frac{1}{\sqrt{2\pi x}} \exp\left(x - \frac{n^2}{2x}\right) \tag{14}$$

of the modified Bessel function I_n reduces [32, 34] the kernel $\mathcal{K}_{n_{32}}$ to

$$\mathcal{K}_{n_{32}}(\gamma, \tilde{\gamma}) \approx \frac{1}{\sqrt{2\pi|\tilde{\gamma}|^2}} \exp\left[-\frac{1}{2}\left(\frac{n_{32}}{|\tilde{\gamma}|} - \frac{\gamma^*\tilde{\gamma} + \gamma\tilde{\gamma}^*}{|\tilde{\gamma}|}\right)^2\right]. \tag{15}$$

Hence, the photon count statistics equation (10) reads

$$W(n_1, n_{32}) = \frac{1}{\sqrt{2\pi|\alpha|^2}} \int d^2\beta\; P(\beta)\; W_{n_1}\left(-\frac{\beta}{\sqrt{2}}\right) \times \exp\left[-\frac{1}{2}\left(\frac{n_{32}}{|\alpha|} - \frac{\alpha^*\beta + \alpha\beta^*}{\sqrt{2}|\alpha|}\right)^2\right]. \tag{16}$$

This result becomes more transparent when we recall the relationship [32]

$$\mathcal{K}_{n_{32}}(\gamma, \tilde{\gamma}) \approx \frac{1}{\sqrt{2}|\tilde{\gamma}|} \left|\left\langle \mathcal{E}_\theta = \frac{n_{32}}{\sqrt{2}|\tilde{\gamma}|} \,\middle|\, \gamma \right\rangle\right|^2 \tag{17}$$

between the homodyne kernel $\mathcal{K}_{n_{32}}$ and the eigenstates $|\mathcal{E}_\theta\rangle$ of the rotated electric field operator

$$\hat{\mathcal{E}}_\theta \equiv \frac{1}{\sqrt{2}}\left(\hat{a}\, e^{-i\theta} + \hat{a}^\dagger\, e^{i\theta}\right). \tag{18}$$

Indeed, for a fixed angle θ of the local oscillator, that is, for $\alpha \equiv |\alpha| e^{i\theta}$, the count statistics given by equation (10) takes the form

$$W(n_1, n_{32}; \theta) = \frac{1}{\sqrt{2}|\alpha|} \int d^2\beta\; P(\beta) \left|\left\langle n_1 \,\middle|\, -\frac{\beta}{\sqrt{2}}\right\rangle\right|^2 \times \left|\left\langle \mathcal{E}_\theta = \frac{n_{32}}{\sqrt{2}|\alpha|} \,\middle|\, \frac{\beta}{\sqrt{2}}\right\rangle\right|^2. \tag{19}$$

This expression is the result of the projection of a photon number state $|n_1\rangle$ and an electric field eigenstate $|\mathcal{E}_\theta\rangle$ on the two output modes of a single beam splitter described by the density operator

$$\hat{\rho}_{\mathrm{out}} \equiv \int d^2\beta\; P(\beta) \left|-\frac{\beta}{\sqrt{2}}\right\rangle_1 \left\langle -\frac{\beta}{\sqrt{2}}\right| \otimes \left|\frac{\beta}{2}\right\rangle_2 \left\langle \frac{\beta}{2}\right|. \tag{20}$$

Consequently, in the strong local oscillator limit we can replace—at least for the purpose of calculation—the beam splitter arrangement on the left-hand side of figure 1 by a device that projects on electric field eigenstates and photon number states as indicated on the right-hand side of the figure.

3.3. Average count rate

We conclude our discussion of the joint intensity–electric field measurement using a six-port interferometer in the homodyne mode by calculating the average count rate

$$s_0 \equiv \sum_{n_1} \sum_{n_{32}} n_1 n_{32} W(n_1, n_{32}) \tag{21}$$

that is the average number of quanta in mode 1 and in the difference n_{32}. This quantity is measured in the experiments of [23, 24].

One way to proceed consists of substituting the expression equation (10) for the photon statistics together with equations (6) and (12) into the definition equation (21) of s_0 and performing the sums. However, it is more convenient to recall the definition equation (9) of $W(n_1, n_{32})$ in terms of the count statistics $W(n_1, n_2, n_3)$, which yields

$$s_0 = \sum_{n_1} \sum_{n_{32}} n_1 n_{32} \sum_{n_2} W(n_1, n_2, n_{32} + n_2) \tag{22}$$

or

$$s_0 = \sum_{n_1} \sum_{n_2} \sum_{n_3} n_1 (n_3 - n_2)\, W(n_1, n_2, n_3)\,. \tag{23}$$

When we substitute the photon distribution $W(n_1, n_2, n_3)$, equation (5), into this expression and recall the first moment

$$\sum_m m\, W_m(\gamma) = |\gamma|^2 \tag{24}$$

of a Poisson distribution together with the normalization property

$$\sum_m W_m = 1, \tag{25}$$

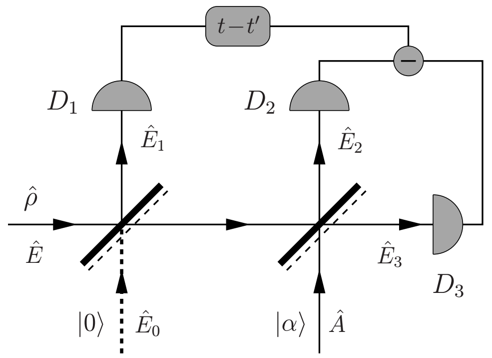

Figure 2. The six-port interferometer used, for example, in [23, 24] to analyse the wave–particle character of light is a generalization of the Hanbury-Brown and Twiss interferometer. It correlates the photon counts at detector D_1 at time t with the difference of counts at the detectors D_2 and D_3 at time t'. The fields at the three input modes are in a state described by the density operator $\hat{\rho}$, in the vacuum state and in a coherent state $|\alpha\rangle$. The corresponding electric field operators are $\hat{E}$, $\hat{E}_0$ and $\hat{A}$, respectively. The operators of the output modes impinging on the three detectors are $\hat{E}_1$, $\hat{E}_2$ and $\hat{E}_3$.

we arrive at

$$s_0 = \frac{1}{2\sqrt{2}} \int \mathrm{d}^2\beta \; P(\beta)\, |\beta|^2 \left(\alpha^* \beta + \alpha \beta^*\right). \tag{26}$$

It is interesting to note that this derivation holds true for any value of $|\alpha|$. In particular, it does not rely on the strong local oscillator limit. It is therefore not surprising that when we use the strong local oscillator limit, equation (16), of the count statistics and evaluate s_0 given by equation (21) by replacing the summation over n_{32} by integration, we also arrive at equation (26).

4. Correlation function $g^{(1.5)}$

Correlation functions of intensities (see for example [33]) reveal quantum mechanical features of the electromagnetic radiation. Since the operator of intensity is the product of two field operators, these correlation functions must always contain an even number of field operators. The most prominent example is the correlation function $g^{(2)}$, which expresses the correlations between two intensity measurements and, therefore, involves four electric field operators. This quantity is at the very heart of the Hanbury-Brown and Twiss [35] experiment, which can be arguably considered the birth of quantum optics.

We now show that the average count rate of the joint intensity–electric field measurement discussed in the preceding section can be described by correlation functions between *three* field operators giving rise to $g^{(1.5)}$. This unusual type of correlation function arises from the fact that one of the fields is in a coherent state. In our treatment, we even allow a time delay between the photon and the electric field measurement.

4.1. Measured signal expressed by $g^{(2)}$

The six-port interferometer of figure 2 correlates the intensity measurement at time t at detector D_1 with the homodyne measurement at time t' at detectors D_2 and D_3. The resulting signal [33]

$$s(t,t') \equiv \left\langle \hat{E}_1^{(-)}(t) \left[\hat{E}_3^{(-)}(t') \hat{E}_3^{(+)}(t') - \hat{E}_2^{(-)}(t') \hat{E}_2^{(+)}(t') \right] \hat{E}_1^{(+)}(t) \right\rangle \tag{27}$$

is the expectation value of the normally ordered product of four field operators. They are the positive and negative frequency components $\hat{E}_j^{(+)}$ and $\hat{E}_j^{(-)}$ of the field operator $\hat{E}_j$ corresponding to the jth mode where $j = 1, 2, 3$.

Since the expectation value is taken with respect to the input states, it is necessary to express the field operators of the output modes in terms of the ones of the input modes. For this purpose, we consider the action of the two beam splitters in figure 2 on the field operators.

A beam splitter represents a linear transformation of field operators [32]. Hence, the operators $\hat{E}_1$, $\hat{E}_2$ and $\hat{E}_3$ of the three exit modes are linear combinations of the three operators $\hat{E}_0$, $\hat{E}$ and $\hat{A}$ of the three input modes corresponding to the vacuum, the field to be investigated and the local oscillator, respectively. When we take into account a phase shift of π in the reflection from the thicker medium, we find for 50:50 beam splitters the combination

$$\hat{E}_1 = \frac{1}{\sqrt{2}} (\hat{E}_0 - \hat{E}) \tag{28}$$

for the field operator $\hat{E}_1$ of mode 1 and the combinations

$$\hat{E}_2 = -\frac{1}{2} (\hat{E}_0 + \hat{E}) + \frac{1}{\sqrt{2}} \hat{A} \tag{29}$$

and

$$\hat{E}_3 = \frac{1}{2} (\hat{E}_0 + \hat{E}) + \frac{1}{\sqrt{2}} \hat{A} \tag{30}$$

for the field operators $\hat{E}_2$ and $\hat{E}_3$ for modes 2 and 3 forming the homodyne detector.

We substitute the field operators $\hat{E}_1$, $\hat{E}_2$ and $\hat{E}_3$ given by equations (28), (29) and (30) into equation (27) and recall that vacuum expectation values of normally ordered products of field operator vanish. As a result we can drop the electric field operator $\hat{E}_0$ of the vacuum mode, which yields

$$\begin{aligned} s(t,t') = \frac{1}{8} \langle \hat{E}^{(-)}(t) \{ & [\hat{E}^{(-)}(t') + \sqrt{2}\,\hat{A}^{(-)}(t')] \\ & \times [\hat{E}^{(+)}(t') + \sqrt{2}\,\hat{A}^{(+)}(t')] \\ & - [-\hat{E}^{(-)}(t') + \sqrt{2}\,\hat{A}^{(-)}(t')] \\ & \times [-\hat{E}^{(+)}(t') + \sqrt{2}\,\hat{A}^{(+)}(t')] \} \hat{E}^{(+)}(t) \rangle, \end{aligned} \tag{31}$$

or

$$\begin{aligned} s(t,t') = \frac{1}{2\sqrt{2}} \langle \hat{E}^{(-)}(t) [& \hat{E}^{(-)}(t') \hat{A}^{(+)}(t') \\ & + \hat{A}^{(-)}(t') \hat{E}^{(+)}(t')] \hat{E}^{(+)}(t) \rangle. \end{aligned} \tag{32}$$

This expression can be interpreted as the sum of two second-order correlation functions $g^{(2)}$. The first one has two negative frequency parts of $\hat{E}$ and one positive frequency part of $\hat{E}$ and $\hat{A}$, whereas the second one has two positive frequency parts of $\hat{E}$ and one negative frequency part of $\hat{E}$ and $\hat{A}$.

4.2. Local oscillator expectation value

Since the local oscillator is in a coherent state $|\alpha\rangle$, we can now simplify equation (32) using

$$\langle \hat{A}^{(\pm)}(t')\rangle = \mathcal{A}_0\,|\alpha|\,\mathrm{e}^{\pm\mathrm{i}\theta}\,\mathrm{e}^{\mp\mathrm{i}\omega t'}. \tag{33}$$

We choose the constant $\mathcal{A}_0$ real and combine the phases appearing in the field operator decomposition with the phase of the coherent state in the phase θ. Therefore, the signal $s(t,t')$ can be cast in the form

$$s(t,t') = \frac{\mathcal{A}_0\,|\alpha|}{2\sqrt{2}}\langle \hat{E}^{(-)}(t)\,[\hat{E}^{(-)}(t')\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{e}^{-\mathrm{i}\omega t'} + \hat{E}^{(+)}(t')\,\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{e}^{\mathrm{i}\omega t'}]\hat{E}^{(+)}(t)\rangle, \tag{34}$$

or

$$s(t,t') = \frac{\mathcal{A}_0\,|\alpha|}{2\sqrt{2}}[\langle \hat{E}^{(-)}(t)\,\hat{E}^{(-)}(t')\,\hat{E}^{(+)}(t)\rangle \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{e}^{-\mathrm{i}\omega t'} + \langle \hat{E}^{(-)}(t)\,\hat{E}^{(+)}(t')\,\hat{E}^{(+)}(t)\rangle \mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{e}^{\mathrm{i}\omega t'}]. \tag{35}$$

Hence, the signal is given by the sum of correlation functions which involve three field operators. In this sense, we can speak of a correlation function $g^{(1.5)}$.

However, we emphasize that the signal s defined by equation (27) involves the product of *four* field operators. Since one of the fields is in a coherent state, the corresponding field operator reduces to a c-number, which we can factor out of the expectation value and arrive at the product of *three* operators.

4.3. Connection to average counts

In order to connect equation (34) with the joint count statistics equation (19), we introduce the quadrature field operator

$$\hat{E}_\theta(t') \equiv \frac{1}{\sqrt{2}}(\hat{E}^{(+)}(t')\,\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{e}^{\mathrm{i}\omega t'} + \hat{E}^{(-)}(t')\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{e}^{-\mathrm{i}\omega t'}) \tag{36}$$

and equation (34) reduces to

$$s(t,t') = \frac{\mathcal{A}_0\,|\alpha|}{2}\langle \hat{E}^{(-)}(t)\,\hat{E}_\theta(t')\,\hat{E}^{(+)}(t)\rangle. \tag{37}$$

This expression brings out most clearly that the measurement summarized in figure 1 measures the correlation between the intensity at time t and the quadrature of the electric field at time t'.

We conclude by showing that for $t=t'$ the signal $s(t,t')$ reduces (up to a constant) to equation (26). For this purpose we recall that the field operators $\hat{E}^{(+)}$ and $\hat{E}^{(-)}$ are proportional to the annihilation operator and the creation operator $\hat{a}$ and $\hat{a}^\dagger$ and obtain from equation (35)

$$s(t,t) = \frac{\mathcal{A}_0\,|\alpha|\,\mathcal{E}_0^3}{2\sqrt{2}}\left[\mathrm{e}^{\mathrm{i}\phi}\langle \hat{a}^\dagger\hat{a}^\dagger\hat{a}\rangle + \mathrm{e}^{-\mathrm{i}\phi}\langle \hat{a}^\dagger\hat{a}\hat{a}\rangle\right]. \tag{38}$$

Here we have combined various phases into a phase ϕ. An additional real constant $\mathcal{E}_0$ appears since we have expressed the field operators in terms of creation and annihilation operators.

Next we use equation (2) to express the density operator $\hat{\rho}$ for the field $\hat{E}$ in terms of the P-distribution. When we take into account that normally ordered expectation values can easily be calculated [33] with the help of the P-distribution, we immediately arrive (up to a constant) at equation (26) with $\alpha \equiv |\alpha|\,\mathrm{e}^{\mathrm{i}\phi}$.

5. Summary

In this paper, we have analysed a simultaneous joint intensity–electric field measurement using a six-port interferometer. For this purpose, we have first used the familiar beam splitter transformation of state vectors to derive an analytical expression for the complete count statistics. We have then focused on the homodyne measurement. In this case the count statistics is given by a phase-space integral of the product of the P-distribution of the state of interest, a Husimi Q-function and the homodyne kernel. In the limit of a strong local oscillator, the kernel reduces to the electric field distribution of a coherent state. As a result, this specific six-port interferometer consisting of two beam splitters acts as a single one with the vacuum and the state of interest in the input modes. The count statistics is then determined by the projection of the two output modes on a photon number and an electric field eigenstate.

It is interesting to compare and contrast this study with one that is based on the beam splitter transformation of the electric field operators. In this way we have obtained an expression for the corresponding correlation function of the six-port interferometer operated in a homodyne mode. Since one of the field modes is in a coherent state, the correlation function describing this setup and containing the product of four field operators reduces to one consisting of three operators giving rise to a correlation function $g^{(1.5)}$. Our study allows for a time delay between the intensity and the field measurement. However, it can only provide the average count rate.

In our analysis of the six-port interferometer, we have concentrated purely on the conceptual aspects of this device. We have neglected any details of or imperfections in the various experimental realizations. Our goal in this effort was to gain deeper insight into, that is, in-depth understanding of, the subtle aspects of the wave–particle duality of light. We do realize that light is one of the essential ingredients of communication between human beings and builds the bridge between science and art. We are reminded of the striking comparison [36] between science and art by the German philosopher Ernst Cassirer (1874–1945):

> ‘There is a conceptual depth as well as a purely visual depth. The first is discovered by science; the second is revealed in art. The first aids us in understanding the reasons of things; the second in seeing their forms. In science we try to trace phenomena back to their first causes, and to general laws and principles. In art we are absorbed in their immediate appearance, and we enjoy this appearance to the fullest extent in all its richness and variety. Here we are not concerned with the uniformity of laws but with the multiformity and diversity of intuitions.’

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