Components of a Cognitive Theory for Education: Implications for the Use of Math in Physics Courses

Edward F. Redish
Department of Physics
University of Maryland
and University of Colorado/JILA

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The UMd PERG:

- **Faculty**
  - Joe Redish (Phys)
  - David Hammer (Phys/C&I)
- **Research Faculty**
  - Andy Elby (Phys)
  - Rachel Scherr (Phys)
- **Postdoc**
  - Ayush Gupta (Phys)

Physics
Curriculum & Instruction
Both

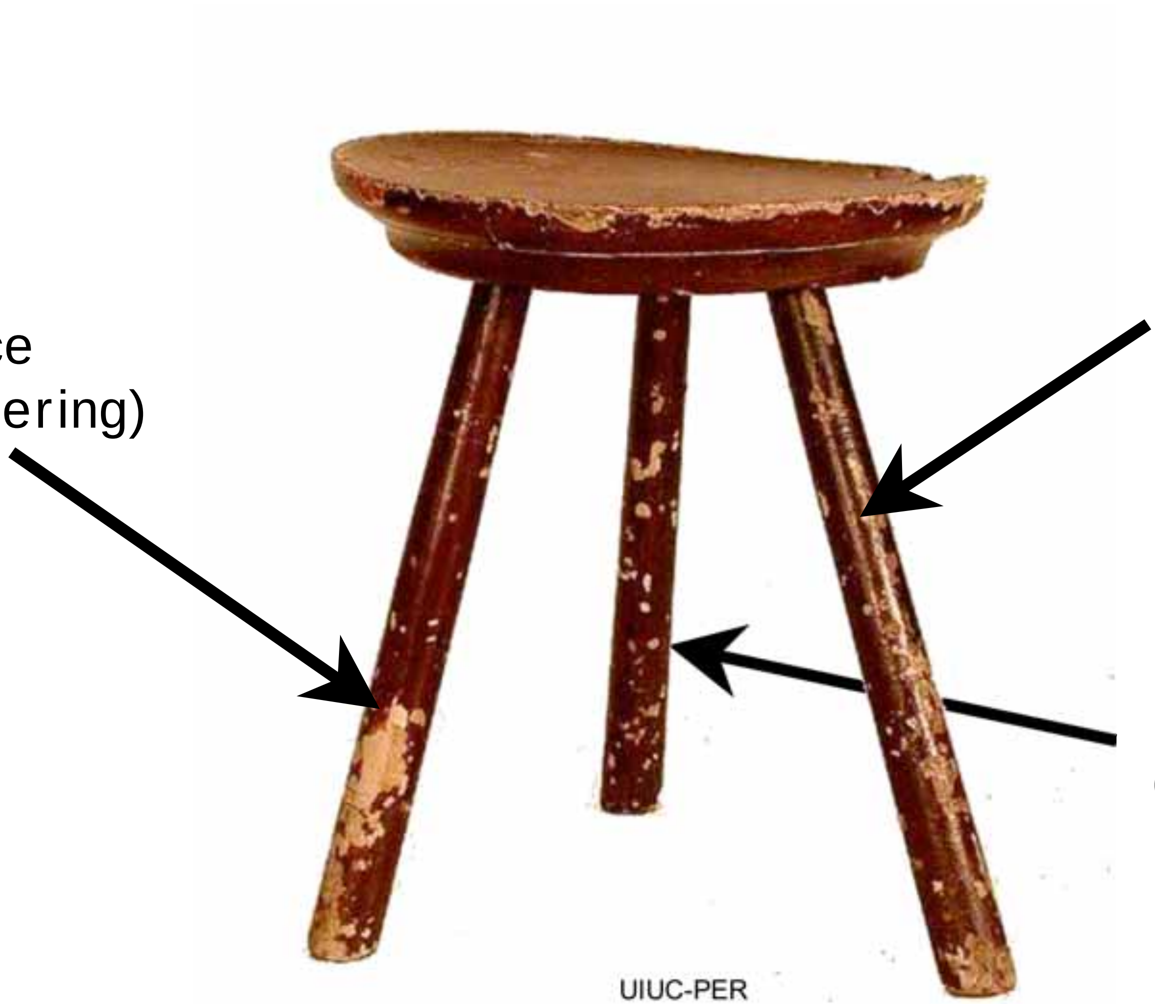
- **Grad Students**
  - Tom Bing (Phys)
  - Luke Conlin (C&I)
  - Brian Frank (Phys)
  - Renee Goertzen (Phys)
  - Ray Hodges (Phys)
  - Paul Hutchinson (C&I)
  - Mattie Lau (C&I)
  - Tim McCaskey (Phys)
  - [Jonathan Tuminaro (Phys)]

[…] = alumnus

What have we learned in PER?

- Constructivism –
  - Students build new knowledge by interpreting new information in terms of what they know.
- Misconceptions –
  - What students bring in to a physics class can lead them to misinterpret what they are supposed to be learning.
- Active learning –
  - Traditional “passive” environments are not as effective as research-based “active engagement” environments.

“The most important leg of a three-legged stool is the one that’s missing.”



Theory

- A classroom is a highly complex system. There are many components to talk about.
  - The structure and function of the learner
  - The structure and function of the classroom environment (on many levels)
  - The interaction of the learner with those environments.
  - The structure of the knowledge to be learned
    - through the perspective of the structures of the learner and learning environments.

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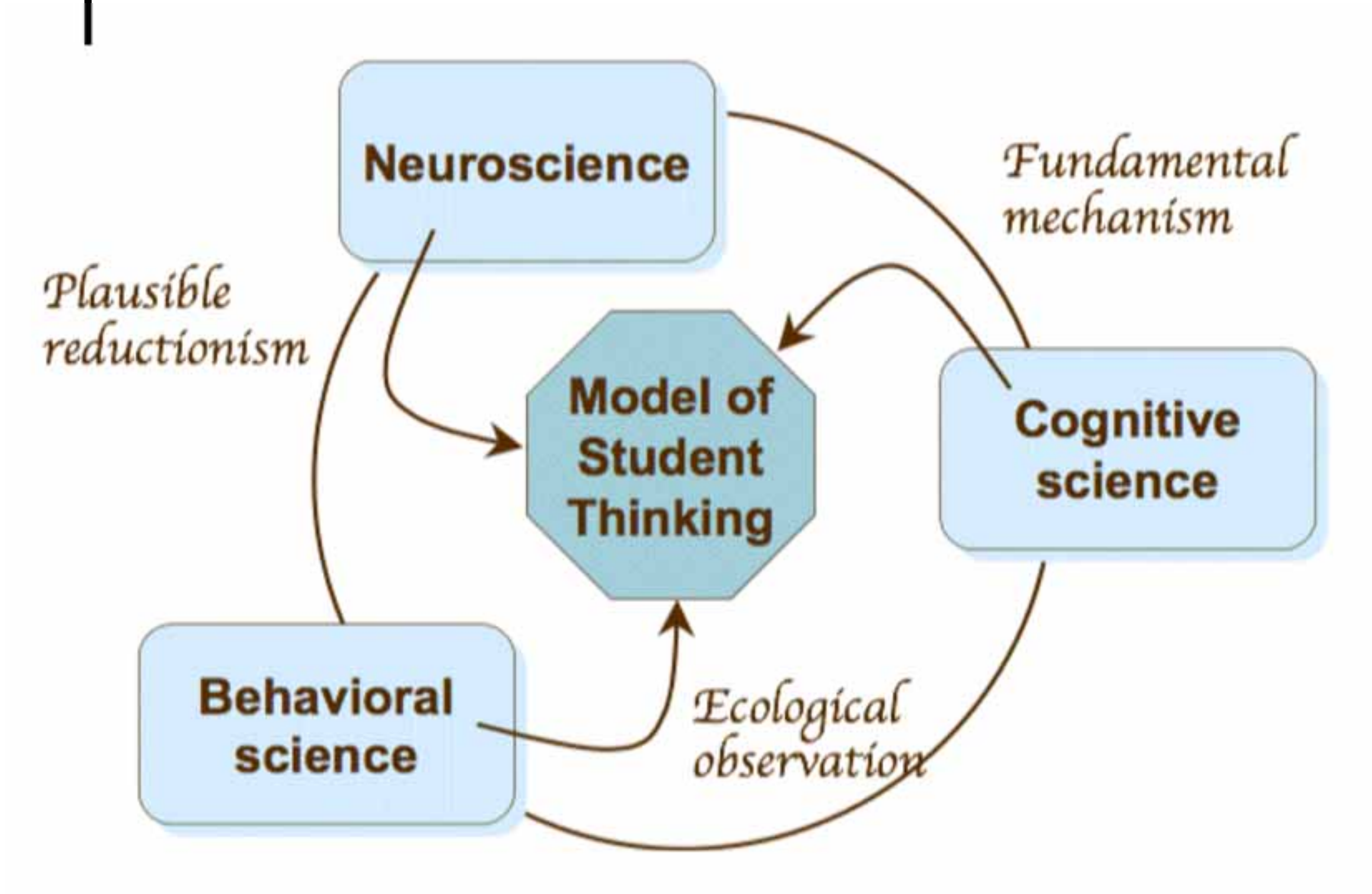
Triangulation



The Resource Framework: Four foothold ideas

1. **Activation / Resources**

   A perception / awareness ("cognit"*) of something corresponds to the activation of a set of linked neurons.

2. **Association**

   The activation of one cognit can lead to the activation of others ("spreading activation")

3. **Binding**

   Different cognits can become tightly tied so they always activate together – the user becomes unaware of their separate parts.

4. **Selective attention / Control**

   Contexts can suppress, prime, or activate clusters of cognits.

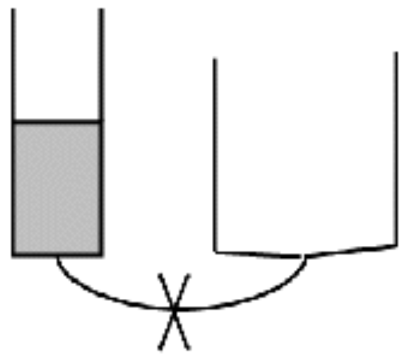
Hammer, Am. J. Phys. Suppl. ***68*** *S52-S59 (2000)*
Redish, Fermi Summer School Lectures (2003)
**Fuster, Memory in the Cerebral Cortex (MIT Press, 1999).*

1. Activation: Resources

- Thinking is dynamic.
- Different knowledge elements or processes (resources) “turn on” and activate other related elements.
- Which related elements are turned on depend on context
- Things that “pop up” can become tightly tied to other elements.

Example

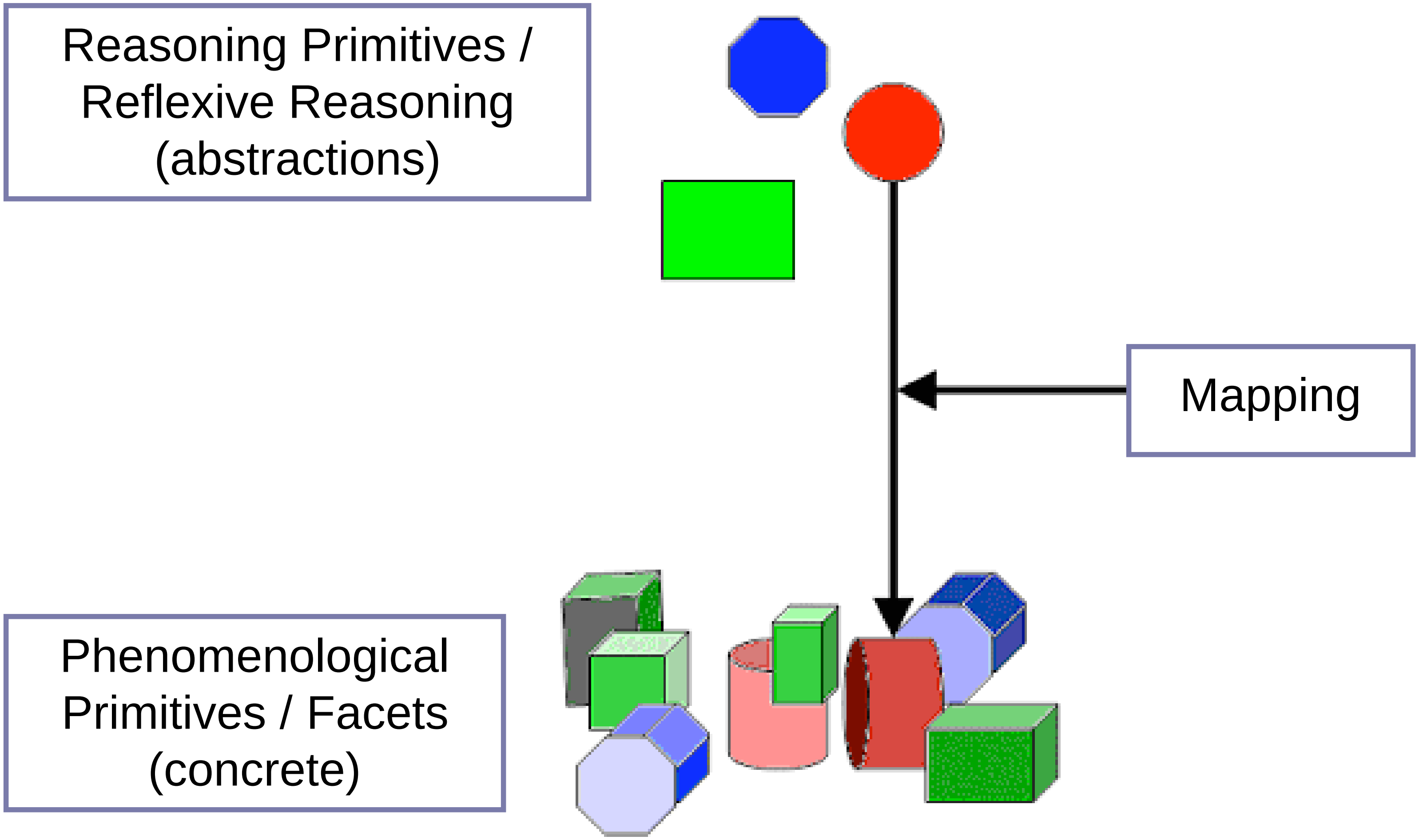
Two containers with water are connected by a rubber tube with a pinch clamp as shown schematically in the figure below. When the clamp is opened, what will happen to the water levels?



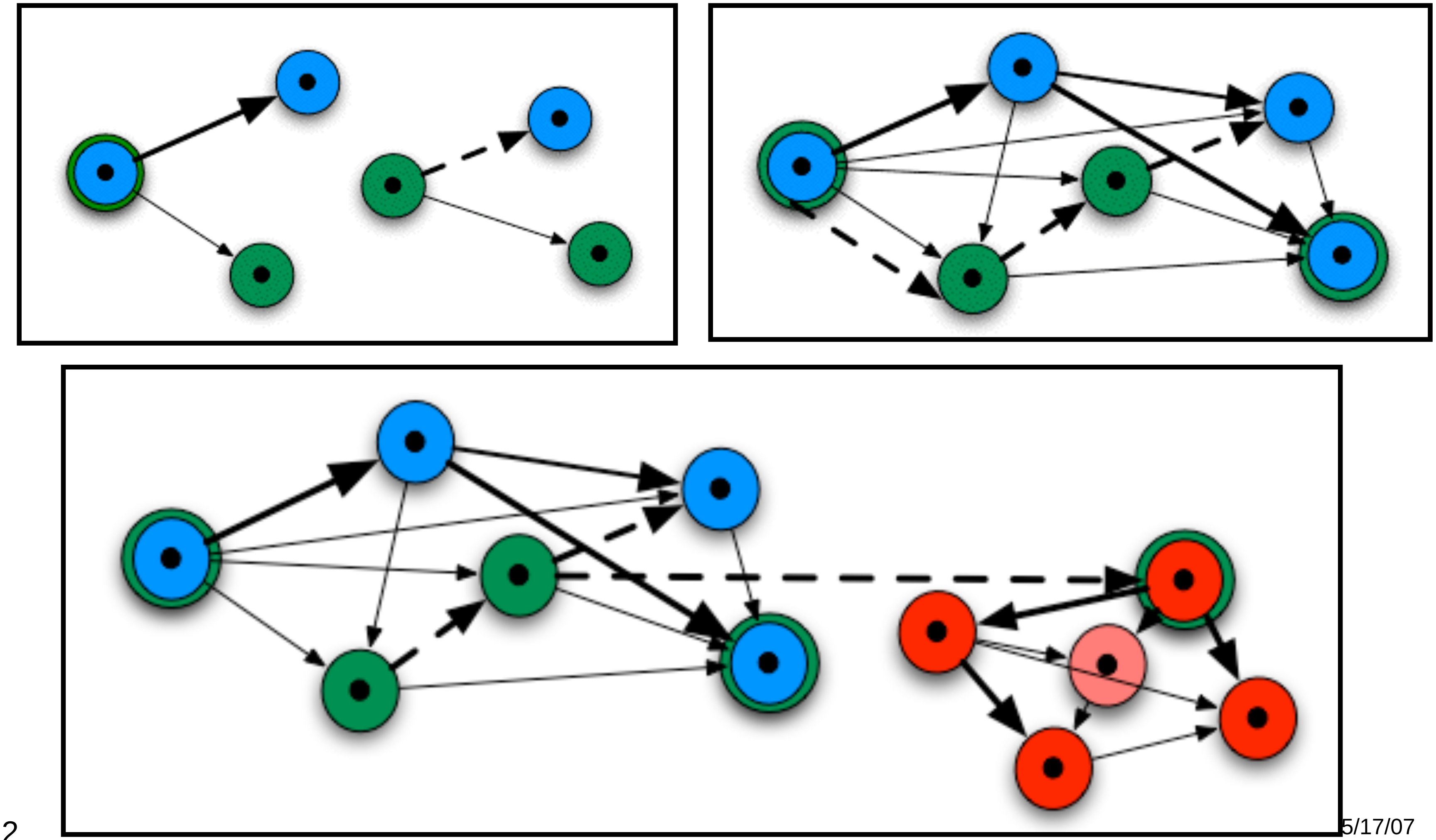
- *It will continue to move until there's some sort of homeostasis, or whatever you call it, an equilibrium, which may be, it's either going to be that the volumes are the same, or the heights will be the same, I'm trying to think of which one it's going to be. Do you want me to tell you which? (laughs) Um, I'll go with volume for now but I'm not sure. So it'd be lower in the larger container….*

M. Loverude interview

Activation Structure



2. Association



Sabella and Redish, Am. J. Phys., to be published (2007)

Associations are fundamental to generating cognitive structures

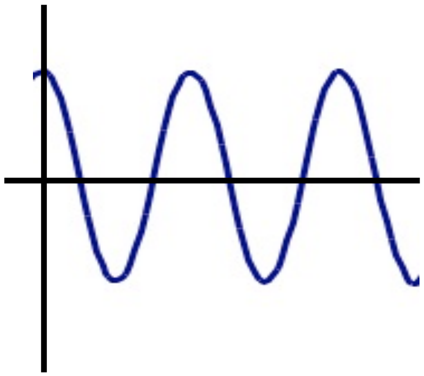
- Used in many areas of cognitive research
  - Excitation / inhibition (neuroscience)
  - Learning / Hebb's rule (neuroscience)
  - Spreading activation (cog sci)
  - Context dependent meaning construction / Encyclopedic knowledge (cog semantics)
- Context dependence
  - Activated associations depend on perceived context (expectations / selective attention)

3. Binding

- As we learn, we bring together many different pieces of knowledge, binding them into a single coherent unit.
- Sometimes this process is very fast, sometimes it takes seconds, sometimes it takes years.
- “Compilation”

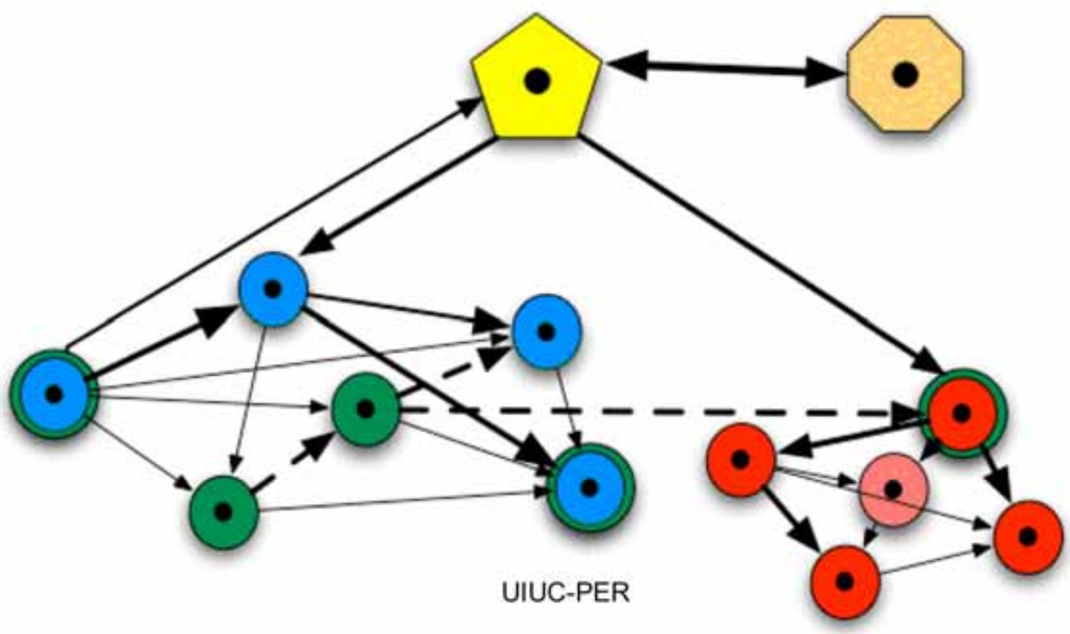
Binding is hard to undo!

- Our processing of visual signals is highly relative and impossible to “unpack”.
- Things we learn can also compile — sometimes over a period of years.
- Can you look at a graph and <u>not</u> immediately know where the derivative is 0?



4. Selective Attention/Control

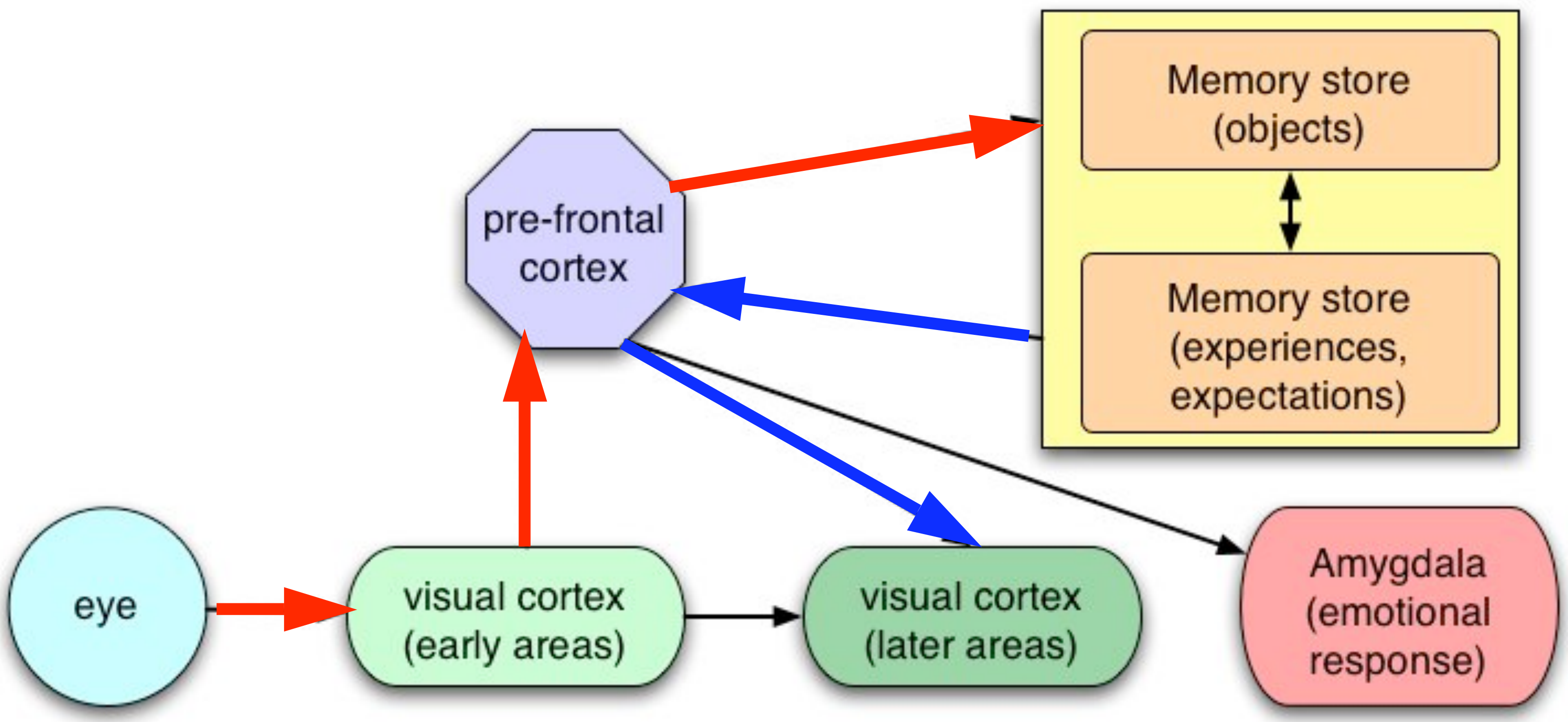
- Synapses can be excitatory or inhibitory.
- The brain is filled with both feedforward links (for association and activation) and feedback links (for switching and context dependence).



Control Structures: How are associations and context dependences controlled?

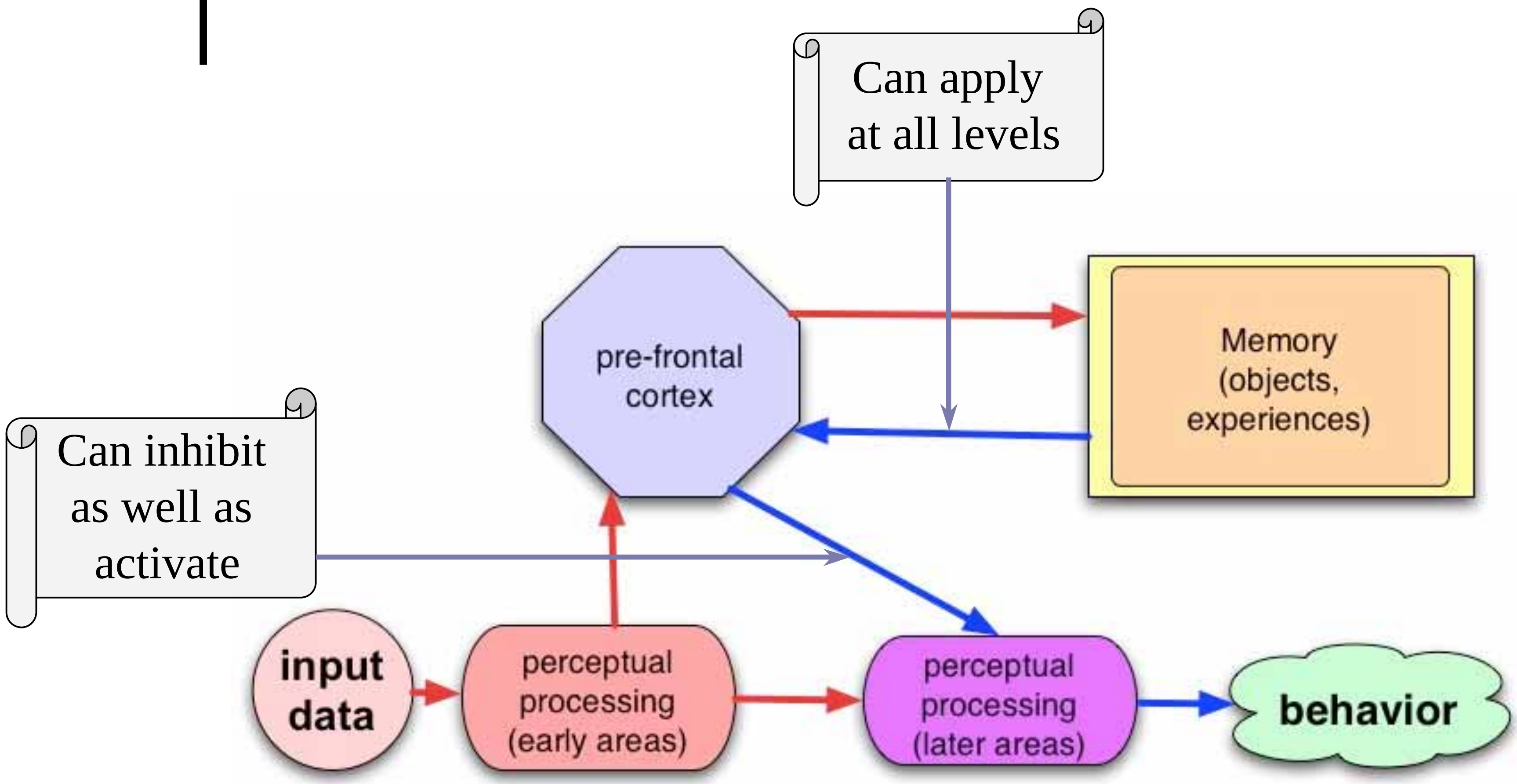
- Neuroscience
  - Top-down facilitation of vision (Moshe Bar)
- Cognitive science
  - Selective attention (Daniel Simons)
- Behavioral science
  - Framing (Irving Goffman, Deborah Tannen)

Neuroscience:
Top-down facilitation



M. Bar, J. Cog. Neurosci. ***15***:*4, 600-609 (2003)*
Fenske, Aminoff, Gronau, & Bar, Prog. in Brain Res. ***155****, 3-21 (2006)*

A general higher order CS



Behavioral Science: Framing

- One way control plays out is through complex selective attention based on social structuring.
- We pay attention to what we expect (often tacitly) is going on and relevant.
- We learn to select and ignore, *framing* our situation — deciding what matters and what doesn't quickly and (often) unconsciously.

I. Goffmann, Frame Analysis (1974)

D. Tannen, Framing in Discourse (1993).

How do the issues of resources, association, binding, and selective attention play out in a physics class?

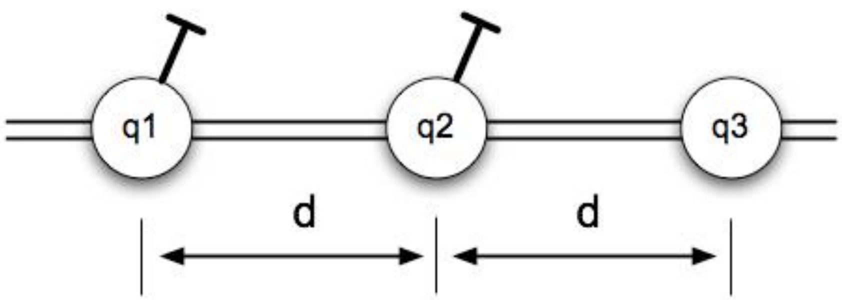
- Binding
  - Group problem solving in algebra-based physics: the 3-charge problem
- Control
  - Group problem solving in algebra-based physics: estimating pressure difference
- Association
  - Individual physics major solving a problem in QM: barrier penetration

Examples: The Data

- *Learning How to Learn Science (2000-2004)*
  - 4-year NSF supported project to study algebra-based physics
  - All parts of the course were modified to
    - increase active engagement
    - focus on epistemological development
    - provide observational data ("ecological")
  - Approximately 1000 hours of videotaped data were collected in lab, tutorial, and HW center.
- *Learning the Language of Science (2005-2009)*
  - 4-year NSF supported project to study use of math in upper division physics
  - So far, about 50 hours of videotapes have been collected of students working on HW.

Binding: A "simple" problem? (algebra-based physics)

- *Three charged particles lie on a straight line and are separated by distances d. q_1 and q_2 are held fixed. q_3 is free to move but is in equilibrium (no net electrostatic force acts on it). If $q_2 = Q$, what value must q_1 have?*



- Setting:
  - Four students working in the course center.

*Redish, Scherr, & Tuminaro, Phys. Teach., **44**, 293 (2006).*

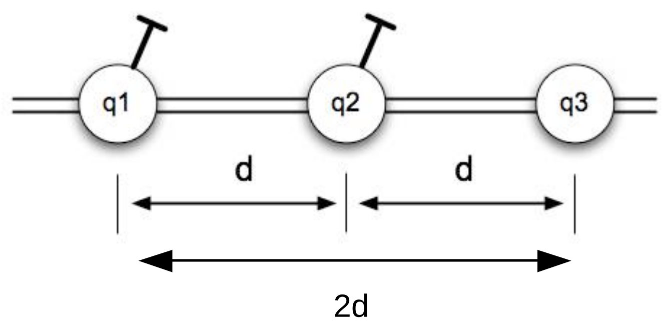
A simple algebra problem – just math?

$$F^{net} = 0 = \frac{k_C q_1 q_3}{(2d)^2} + \frac{k_C q_2 q_3}{(d)^2}$$

$$\frac{k_C q_1 q_3}{(2d)^2} = -\frac{k_C q_2 q_3}{(d)^2}$$

$$\frac{q_1}{4} = -q_2$$

$$q_1 = -4q_2 = -4Q$$



Four students took 45 minutes to solve it.

45 minutes?

- When we first viewed the video we were concerned that they took so long to solve what (on the surface) seemed to be a simple problem.
- After a careful analysis, we became convinced that the work they did was worthwhile and a valuable part of their learning.

How they get there

Description of events
They make some progress thinking qualitatively, but are at first unsure about forces, directions, and fields.
The Teaching Assistant suggests they draw a diagram so they can agree on what is happening.
They now agree on which charges are exerting which forces in which directions and settle on a factor of -2.
One student, recalling a result of the non-linearity in a previous problem tries to get them to think using the equation (Coulomb's law).
Eventually, she manages to turn their attention to using the equation and she works out the correct solution to the problem using algebra — constructing a clean proof. The group is convinced.

Why so long? The professor's "simple" solution involves lots of hidden resources. Our list:

- Like charges repel, unlike attract
- Attractions and repulsions are forces
- Forces can add and cancel (one does not "win"; one is not "blocked")
- "Equilibrium" corresponds to balanced, opposing forces (not a single strong "holding" force)
- Electric force both increases with charge and decreases with distance from charge
- Objects respond to the forces they feel (not those they exert)
- Charges may be of indeterminate sign and still exert balancing forces on the test charge
- "Fixed" objects don't give visible indication of forces acting on them; "free" ones do
- Only forces on the test charge require analysis
- Each other charge exerts one force on test charge
- Each force may be represented by a vector
- "Equilibrium" corresponds to opposing vectors
- Vertical and horizontal dimensions are separable
- One dimension is sufficient for analysis
- Electric force both increases with charge and decreases with distance from charge
- Electric force decreases with the square of the distance

In this case, the students do what I want them to.

- They first make qualitative sense of the problem.
- Then they:
  - – nail down what they remember from their study of Newton's laws
  - – clarify the nature of the electric force
  - – estimate a qualitative result
  - – refine it by applying the quantitative principle – Coulomb's Law (correctly).

Reverse engineering expert knowledge

- I had failed to appreciate how much was compiled into my “simple” solution.
- Watching these students helped me “reverse engineer” what I had built over many years into a tight, automatic knowledge structure.
- The students are not only solving a problem. They are compiling the knowledge required for the problem and are learning how to solve problems in general.
- The fact that they are willing to work for an hour on a “short” problem is notable.

Control: Choosing resources

- The following problem was given in the first semester of an introductory algebra-based physics class.
  - *Estimate the difference in air pressure between the floor and the ceiling in your dorm room. (Note: you may take the density of air to be 1 kg/m^3.)*

- Setting:
  - Four students working in the course center.

$$p = p_0 + \rho g d$$

$$p_{ceiling} = p_0$$

$$p_{floor} = p_0 + \rho g h$$

$$p_{floor} - p_{ceiling} = \rho g h \approx \left(1 \ \frac{\text{kg}}{\text{m}^3}\right)\left(10 \ \frac{\text{N}}{\text{kg}}\right)(3 \text{ m}) = 30 \ \frac{\text{N}}{\text{m}^2} = 30 \text{ P}$$

Student control structures for Problem Solving

- Coherence – an epistemological resource.
  - Just like any resource, it can be “right or wrong” depending on how it is used.
- In watching students solving physics problems we have observed:
  - They tend to work within a locally coherent organizational framework —
    that only employs a fraction
    of their problem solving resources
  - Sometimes this excludes resources
    that would have been useful
    (or even crucial).

An inappropriate path

- Decided she needed an equation for pressure: chose *PV* = *nRT.*
- Decided she needed the volume for the room.
- Decided it must be 1 m^3. (?!)
- Maintained that, despite TA's hint, "I think you'll agree with me this is an estimation problem."
- Decided if it wasn't 1 m^3, then the prof probably gave the value in a previous HW.

E-games: A selective attention to particular associational patterns

- *Epistemic games* *** —
  - a coherent local (in time) pattern of association for building knowledge or solving a problem. May include
    - Entry conditions
    - Allowed (and forbidden) moves
    - Form or product
    - Identifiable goal

> These can be applied normatively – talking about games we think they ought to be playing*
> or ethnographically – describing the games they actually are playing.**

Collins and Ferguson, Educational Psychologist,* ***28* *25 (1993)*
*** Tuminaro and Redish, TBP in Phys. Rev. ST-PER (2007)*

What E-Game?

Professor

- Goal
  - To find a pressure difference.
- Moves
  - Evaluate equation in terms of mechanism
  - Choose equation
  - Estimate numbers from experience
  - Calculate result

Student

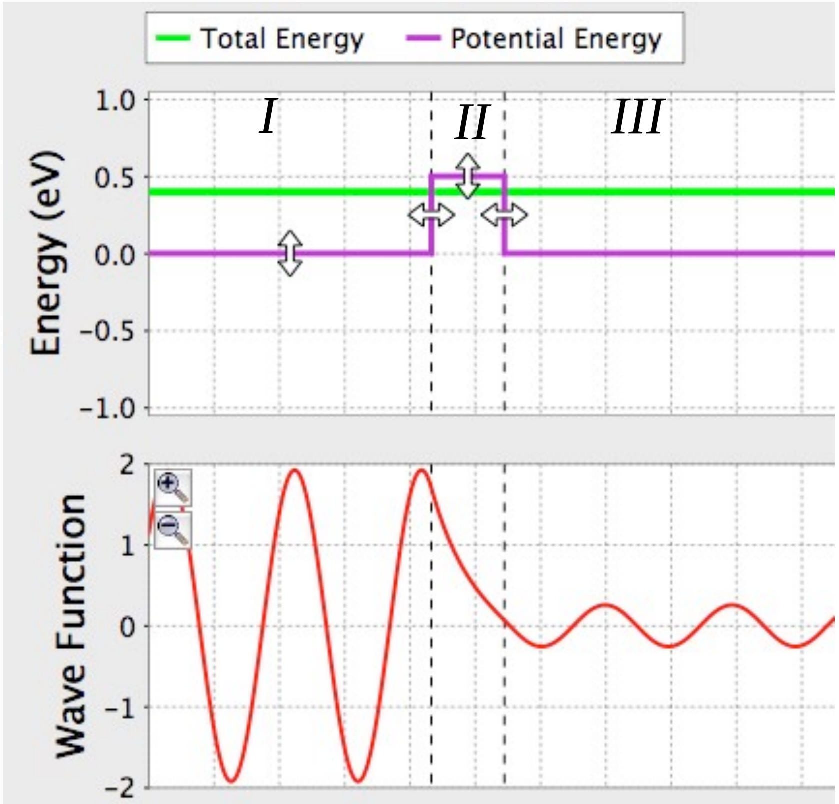
- Goal
  - To find a pressure
- Moves
  - Choose equation
  - Determine what's known
  - Find equation for other unknowns
- Forbidden Move
  - Evaluate equation in terms of mechanism
  - Estimate numbers from experience

The 3-charge students showed strong evidence of e-gaming

- Blocks of time (3-15 min.) were spent using limited reasoning.
- Other appropriate tools were not accessed during these time blocks.
- The students later showed that they both possessed and could use the non-accessed (suppressed?) knowledge effectively.

Association: Undergrad QM

- The following problem was given in the second term of UG QM.
  - *A beam of electrons of energy E is incident on a square barrier of height V_0 and width a. Find the reflection and transmission coefficients, R and T.*
- Setting:
  - One student in an interview setting



*

$$\psi_I = Ae^{ikx} + Be^{-ikx}$$

$$\psi_{II} = Ce^{\kappa x} + De^{-\kappa x}$$

$$\psi_{III} = Ee^{ikx} + Fe^{-ikx}$$

$$k^2 = \frac{2mE}{\hbar^2}$$

$$\kappa^2 = \frac{2m(V-E)}{\hbar^2}$$

* *CU PhET simulation*

$$\psi_I\big|_{x=0} = \psi_{II}\big|_{x=0} \qquad \psi_I'\big|_{x=0} = \psi_{II}'\big|_{x=0}$$

$$\psi_{II}\big|_{x=a} = \psi_{III}\big|_{x=a} \qquad \psi_{II}'\big|_{x=a} = \psi_{III}'\big|_{x=a}$$

4 equations in 6 unknowns

take $F = 0$

churn: solve for B, C, D, E in terms of A.

$$R = \frac{|B|^2}{|A|^2} \qquad T = \frac{|E|^2}{|A|^2}$$

A path to a solution?

- In class, the instructor showed the solutions and the student had copied them down.
- He followed an expected procedure but was unable to recover from minor errors without help.
  - He made some mistakes in copying – keeping the "i's" in the wave function's exponents in region II.
  - He was totally stuck – kept looking through notes and text trying to find the "correct" form. (He later showed that he was easily able to generate the solution from the SE.)
  - When looking at the solution in the negative energy region he immediately claimed – and maintained despite pressure – that there could be no growing exponential term. "We throw that away." (He later was able to work out the solution correctly from the SE.)

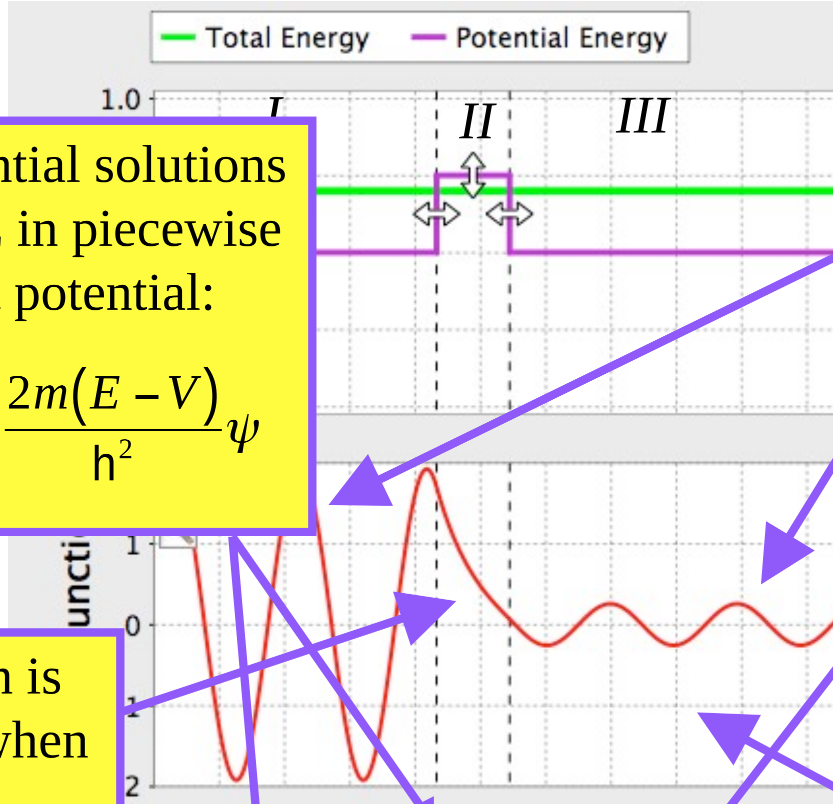
What E-Game?

Professor

- Goal
  - Calculate R and T as function of E, V_0, a.
- Moves
  - Write y in each region.
  - Match y, y' across boundaries.
  - Find currents
  - Solve for R, T.
- Hidden moves
  - Check soln. with SE
  - Check units
  - …

Student

- Goal
  - Calculate R and T as function of E, V_0, a.
- Moves
  - Write y in each region.
  - Match y, y' across boundaries.
  - Find currents
  - Solve for R, T.
- Hidden moves
  - Copy solutions from lecture notes



*

Exponential solutions from SE in piecewise constant potential:

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E-V)}{\hbar^2}\psi$$

Wavefunction oscillates when energy is positive

Change of sign of V-E leads to reversal of character of solutions.

Wavefunction is exponential when energy is negative

Amplitude drops passing through barrier, but λ stays same (because E same on both sides)

$$\psi_I = Ae^{ikx} + Be^{-ikx}$$

$$\psi_{II} = Ce^{\kappa x} + De^{-\kappa x}$$

$$\psi_{III} = Ee^{ikx} + Fe^{-ikx}$$

$$k^2 = \frac{2mE}{\hbar^2}$$

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Growing exponential still present since region is bounded (doesn't go to infinity)

* *CU PhET simulation*

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Continuous wavefunction and derivatives correspond to no infinite potentials.

4 equations in 6 unknowns

take $F = 0$

churn: solve for B, C, D, E in terms of A.

Treatment of solution relies on understanding of meaning of traveling waves.

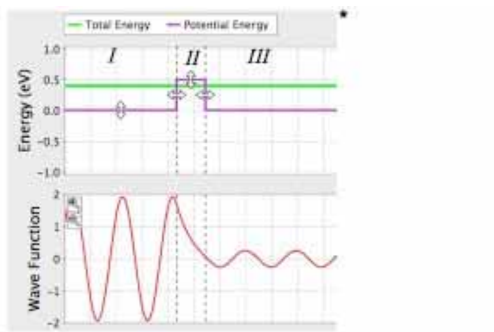
Solutions relative to A, F must be 0.

$$R = \frac{|B|^2}{|A|^2} \qquad T = \frac{|E|^2}{|A|^2}$$

Structure of coefficients depends on understanding of particle current.

Novice vs. Expert

- The novice solution gets the math
- The expert adds a web of physics associations



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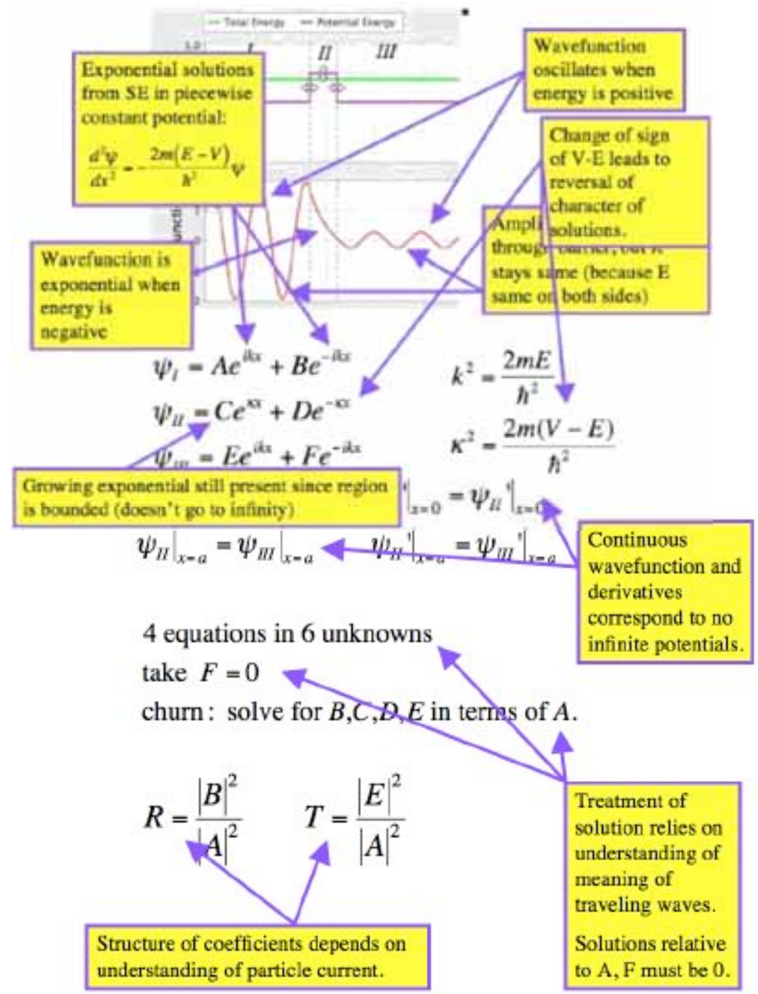


Implications of the Theoretical Framework for the Physics Major

- Our analysis suggests we need to rethink and unpack the structure of our knowledge
- Math plays a large role in the physics major.
- Rethink the role of mathematics in problem solving.
  - Unpack to determine relevant (normative) E-Games
  - Understand what E-Games (ethnographic) students play

Unpacking math in physics

- Math in physics class is not the same as math in a math class
  - We use many different symbols — and not just in the standard “math” ways.
  - We use the same symbol to mean different things, the interpretation depending on context.
  - We blur the distinction between constants and variables depending on the physics.
  - We use equations not just to calculate but to organize our conceptual knowledge.
- But even more important – we put meaning to math differently from in a math class.

Traxoline

Attributed to Judith Lanier

- It is very important that you learn about traxoline. Traxoline is a new form of zionter. It is montilled in Ceristanna. The Ceristannians gristeriate large amounts of fevon and then bracter it to quasel traxoline. Traxoline may well be one of our most lukized snezlaus in the future because of our zionter lescelidge.

Directions: Answer the following questions in complete sentences. Use your best handwriting.

1. What is a traxoline?
2. Where is traxoline montilled?
3. How is traxoline quaselled?
4. Why is it important to know about traxoline?

Math-as-math and math-in-physics are different!

- Our fundamental processing of equations is more complex than in a math class and relies heavily on strong associations with our real-world experience.
  - We associate our interpretation of the equation with a physical system — which lends information on how to interpret the equation
  - We use particular symbols that carry ancillary information not otherwise present in the mathematical structure of the equation
  - We use more complex quantities than in math classes and use them tacitly.

An Example

- If

$$T(x,y) = k(x^2 + y^2)$$

$$where\ k\ is\ a\ constant$$

- then what is

$$T(r,\theta) = ?$$

Which did you choose?

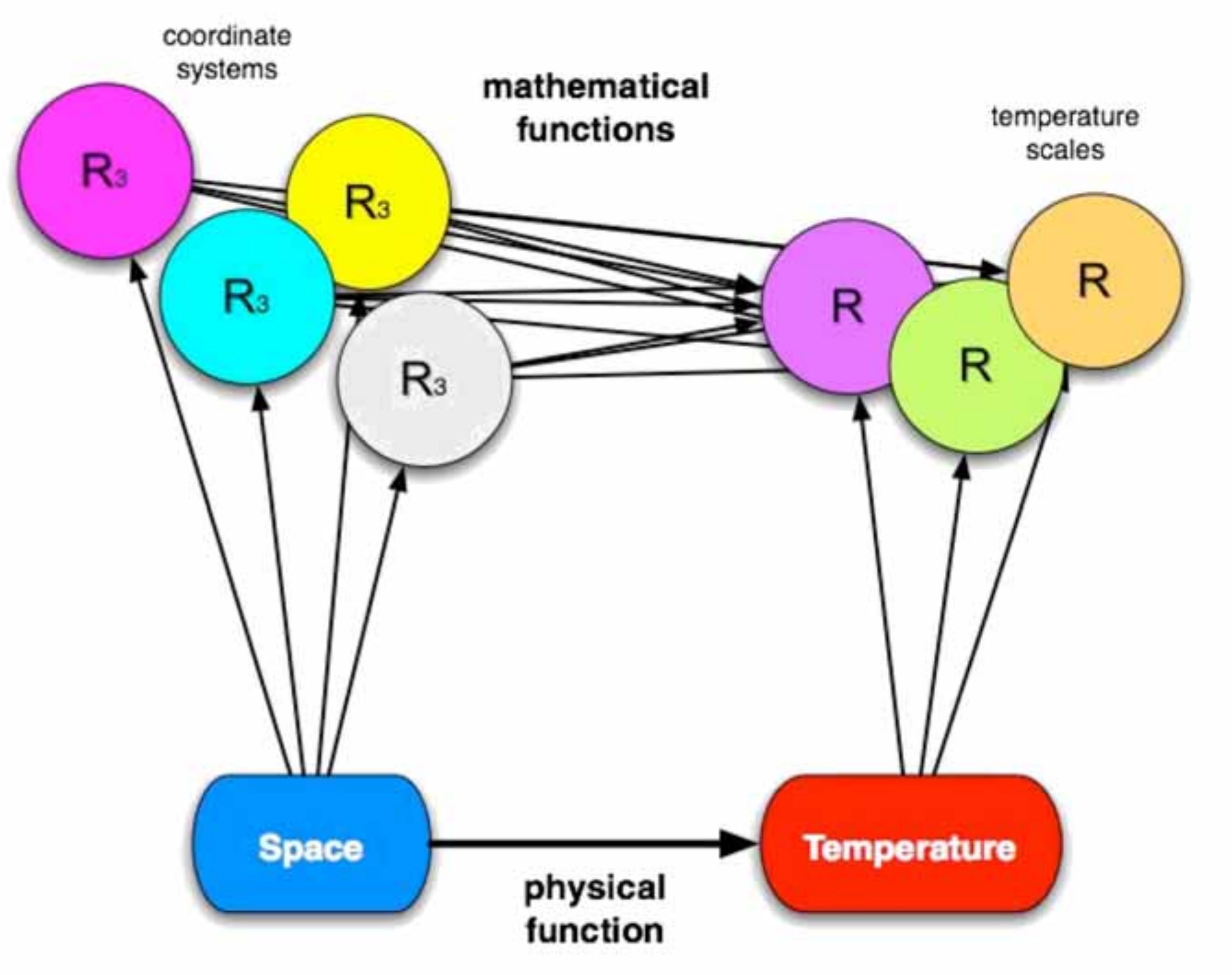
$$A.\quad T(r,\theta) = kr^2$$

$$B.\quad T(r,\theta) = k(r^2 + \theta^2)$$

$$C.\quad T(x,y) = S(r,\theta) = kr^2$$

$D.$ other

Mathematical and physical functions are different

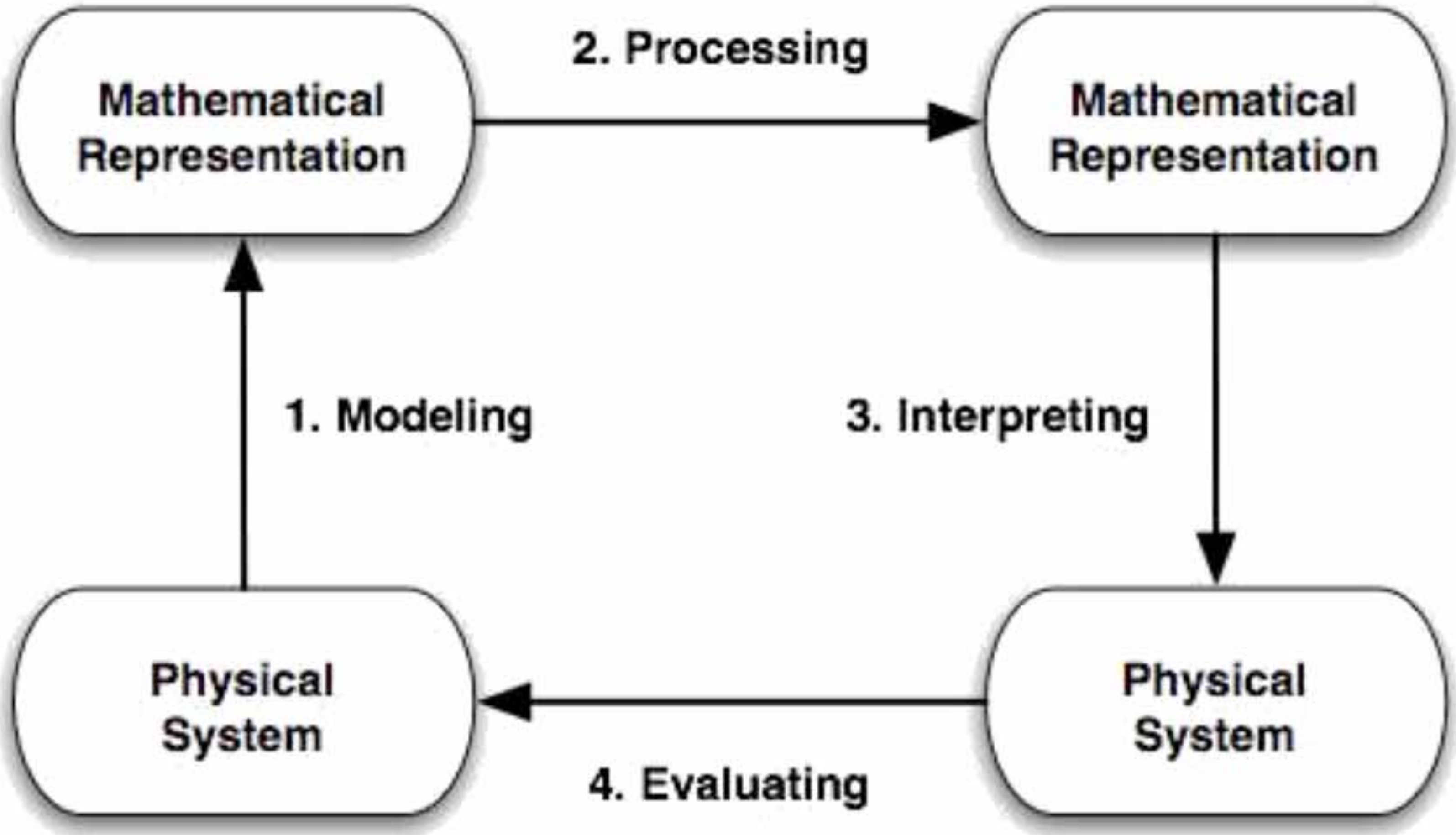


Blending Physics and Math

- Putting together math and physics requires "blending" of two mental spaces to create something new and emergent.
- Our physical knowledge, experience, and intuition creates an associational web of "encyclopedic knowledge" that puts meaning to the math.
- In order to help our students make the transition to this expert level we need to "unpack" how we put them together.

Fauconnier & Turner, The Way We Think: Conceptual Blending (2002)
Evans & Green, Cognitive Linguistics: An Introduction (2006)

Unpacking our use of math in physics



Our traditional approach may not help students focus on critical issues.

- Texts and traditional problems often focus on processing and rarely ask students to model, interpret, or evaluate.
- Instructors may not be able to unpack their expert knowledge and fail to recognize what's complex in a problem.
- Students don't get these ideas in math and may ignore critical associations with physics even if the instructor discusses them.

What can we do to help?

A problem that goes beyond processing

The pair of coupled non-linear ODEs are referred to as the Lotka-Volterra equations and are supposed to represent the evolution of the populations of a predator and its prey in time.
The constants A, B, C, D are positive.

$$\frac{dx}{dt} = Ax - Bxy$$

$$\frac{dy}{dt} = -Cy + Dxy$$

- *Which of the variables, x or y, represents the predator?*
- *Which represents the prey?*
- *What reasons do you have for your choice?*
- *What's left out of this model?*

Conclusion

- “Misconceptions” and “conceptual learning” are only the start to what we need to learn from PER –
  - We want to understand the difficulties with learning problem solving.
  - We want to move to upper division physics.
- Having a theoretical framework helps.
  - It helps in identifying and deconstructing the structure of our own knowledge
  - It helps in identifying the resources the students naturally tend to use and how they organize them.
- We can use the theoretical frame to
  - guide future experiments and observations
  - suggest changes in our curricular materials and environments.