Making Meaning With Math In Physics

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Physics uses a lot of math, but…

- Math in physics class is not the same as math in a math class
  - We use many different symbols — and not just in the standard "math" ways.
  - We use the same symbol to mean different things, the interpretation depending on context.
  - We blur the distinction between constants and variables depending on the physics.
  - We use equations not just to calculate but to organize our conceptual knowledge.
- But even more important – we put meaning to math differently from in a math class.

An example from practice

- A very small charge q is placed at a point somewhere in space. Hidden in the region are a number of electrical charges. The placing of the charge q does not result in any change in the position of the hidden charges. The charge q feels a force, F. We conclude that there is an electric field at the point that has the value $E_0 = F/q$.

  If the charge q were replaced by a charge $-3q$, then the electric field at the point would be

  a) Equal to $-E_0$

  b) Equal to E_0

  c) Equal to $-E_0/3$

  d) Equal to $E_0/3$

  e) Equal to some other value not given here.

  f) Cannot be determined from the information given.

Nearly half of 200 students chose this answer. (→ c)

Given in lecture in algebra-based physics after extensive discussion of electric field. Most students could repeat that E field is independent of the test charge.

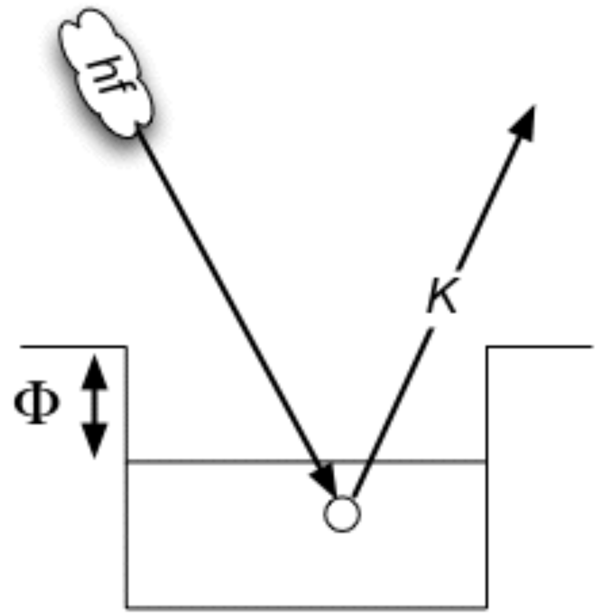
An example from practice

Given on the final exam in engineering physics.

- The photoelectric effect equation is

$$eV_0 = hf - \Phi$$

- If a frequency of f leac to no electrons being emitted, what will happen if we choose a lower frequency?



Nearly ¼ of 160 students answered that you would get electrons and explained that "if the equation gave 0 before and you changed something, it would no longer give 0."

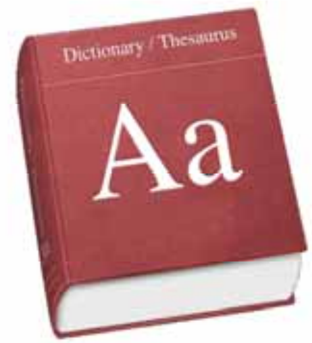
Using math in science needs more than just knowing the math

- The critical part that is missing in pure math class is the blending of physical information with the mathematical.
- This blend often replaces what would be very complex math by intuition from everyday experience.

What does "meaning" mean?

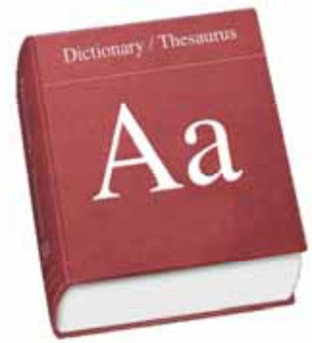
- If we are going to understand
  how we put meaning to math,
  we might start by trying to understand
  how we put meaning to anything.
- Modern cognitive linguists study this topic under the rubric of "cognitive semantics."
  - George Lakoff
  - Ronald Langacker
  - Leonard Talmy
  - Vyvyan Evans ….

Dictionary meaning?



- I do not read arabic, but I know a bit about how an arabic dictionary might work.
  - It would read from right to left.
  - The word to be defined would be on the right of an entry.
- From that, I could find the word "الألكترونات." given enough time.
- But finding it would not help me figure out what it means.
- So how does a dictionary work?

Dictionary meaning!



- Dictionaries are fundamentally circular: words are defined in terms of words.
- The value of a dictionary lies in the hope that as you traverse the circle, you will come upon some set of terms that you already know (and have ultimately learned in some other way than from definitions).

Basic ideas for meaning building

- *Embodied cognition:*
  The meaning of words is grounded in physical experience.
- *Encyclopedic knowledge*:
  Ancillary knowledge is critical in the creation of meaning.
- *Conceptualization*:
  Meaning is constructed dynamically.

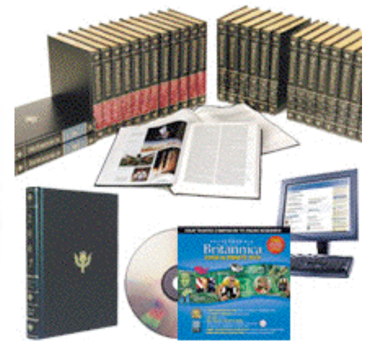
Vyvyan Evans & Melanie Green, *Cognitive Semantics*

Embodied cognition

- In the end, all of our understanding of even complex concepts must come down to direct perceptual experience.
- Many processes enable the building of this extraordinary and complex linguistic structure:
  - Metaphor (Lakoff & Johnson)
  - Polysemy (Langacker, Evans)
  - Blending (Fauconnier & Turner)

Educational researchers will recognize this as a direct (and inevitable) consequence of the principle of Piagetian constructivism.

“Encyclopedic” knowledge



- We interpret the words we hear or read in terms of a vast knowledge and experience of the world.

- (I know: Real encyclopedias are made up of words just like dictionaries – just more of them. Maybe this is not a good term.)

Conceptualization

- Language does not directly code for semantic meaning.
- Rather, linguistic units are prompts for the construction of meaning within a given conceptual / contextual frame.
- This means that meaning is dynamically constructed – a process rather than something fixed and stable.

Meaning means ….?

- "…meaning is not a property of language per se, but rather is a function of language use, and thus, a characteristic of a process of meaning construction, rather than relating to mental entities/units stored in memory."

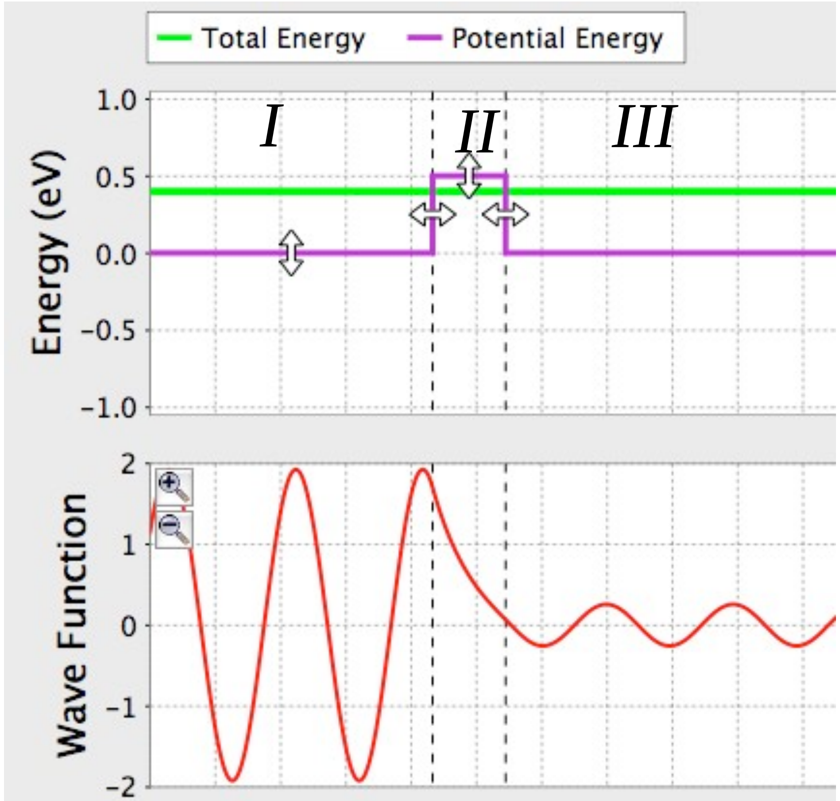
V. Evans, "Lexical Concepts, Cognitive Models and Meaning-Construction,"
Cognitive Linguistics, **17**:4 (2006) 491-534.

Meaning in math?

- These ideas suggest that meaning relies on
  - Organized knowledge structures (frames)
  - Associational patterns and activation (framing)
  - The linking of different kinds of knowledge (blending)
- Applying it to math-in-physics they suggest that meaning derives from the association of math with a physical context and physical ideas.

Example: Undergrad QM

- The following problem was given in the second term of UG QM.
  - A beam of electrons of energy E is incident on a square barrier of height V_0 and width a. Find the reflection and transmission coefficients, R and T.
- The student in this example followed an expected procedure but was unable to recover from minor errors.



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$$\psi_I = Ae^{ikx} + Be^{-ikx}$$

$$\psi_{II} = Ce^{\kappa x} + De^{-\kappa x}$$

$$\psi_{III} = Ee^{ikx} + Fe^{-ikx}$$

$$k^2 = \frac{2mE}{\hbar^2}$$

$$\kappa^2 = \frac{2m(V-E)}{\hbar^2}$$

* CU PhET simulation

$$\psi_I\big|_{x=0} = \psi_{II}\big|_{x=0} \qquad \psi_I'\big|_{x=0} = \psi_{II}'\big|_{x=0}$$

$$\psi_{II}\big|_{x=a} = \psi_{III}\big|_{x=a} \qquad \psi_{II}'\big|_{x=a} = \psi_{III}'\big|_{x=a}$$

4 equations in 6 unknowns

take $F = 0$

churn: solve for B, C, D, E in terms of A.

$$R = \frac{|B|^2}{|A|^2} \qquad T = \frac{|E|^2}{|A|^2}$$

Meaning arising from context

- In lecture, the instructor showed the solutions in each region and the student had copied them down.
- He made some mistakes in copying – keeping the "i's" in the wave function's exponents in region II.
- He was totally stuck – kept looking through notes and text trying to find the "correct" form.
- He later showed that he was easily able to generate the solution from the SE.

What E-Game?

Professor

- Goal
  - Calculate R and T as function of E,V_0, a.
- Moves
  - Write ψ in each region.
  - Match ψ and ψ' across boundaries.
  - Find currents
  - Solve for R,T.
- Hidden moves
  - Check soln. with SE
  - Check units
  - Apply physical constraints
  - …

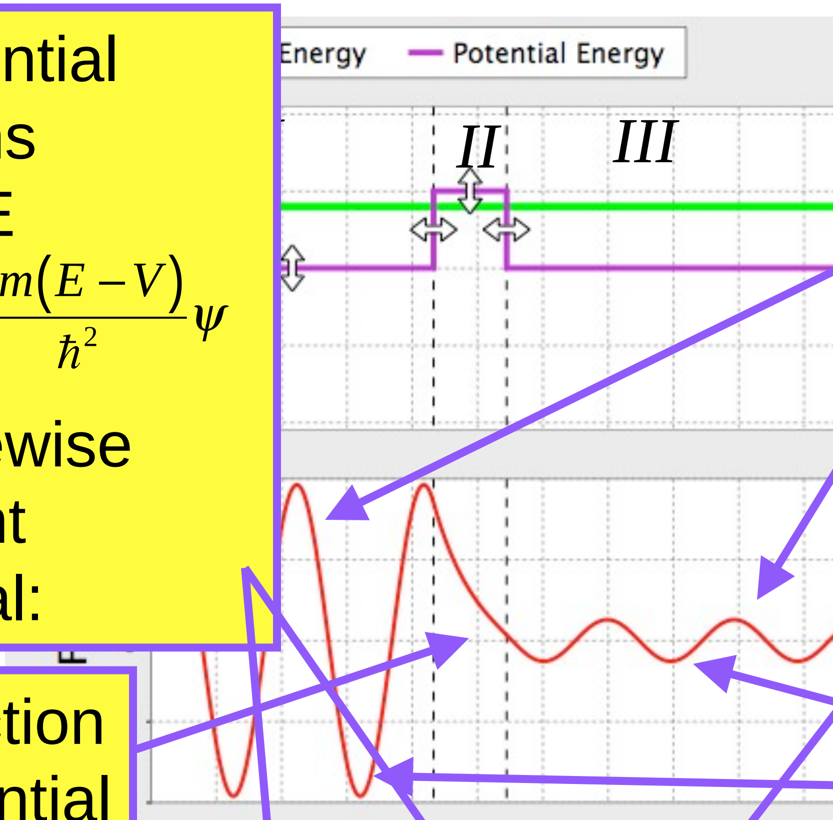
Student

- Goal
  - Calculate R and T as function of E,V_0, a.
- Moves
  - Write ψ in each region.
  - Match ψ and ψ' across boundaries.
  - Find currents
  - Solve for R,T.
- Hidden moves
  - Copy solutions from lecture notes

Exponential solutions from SE

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E-V)}{\hbar^2}\psi$$

in piecewise constant potential:



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Wavefunction oscillates when energy is positive

Change of sign of V-E leads to reversal of character of solutions.

Wavefunction is exponential when energy is negative

$$\psi_I = Ae^{ikx} + Be^{-ikx}$$

$$\psi_{II} = Ce^{\kappa x} + De^{-\kappa x}$$

$$k^2 = \frac{2mE}{\hbar^2}$$

$$\kappa^2 = \frac{2m(V-E)}{\hbar^2}$$

Amplitude drops passing through barrier, but λ stays same (because E same on both sides)

Growing exponential still present since region is bounded (doesn't go to infinity)

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$$\psi_I\big|_{x=0} = \psi_{II}\big|_{x=0} \qquad \psi_I'\big|_{x=0} = \psi_{II}'\big|_{x=0}$$

$$\psi_{II}\big|_{x=a} = \psi_{III}\big|_{x=a} \qquad \psi_{II}'\big|_{x=a} = \psi_{III}'\big|_{x=a}$$

Continuous wavefunction and derivatives correspond to no infinite potentials.

4 equations in 6 unknowns

take $F = 0$

churn: solve for B, C, D, E in terms of A.

Treatment of solution relies on understanding of meaning of traveling waves.

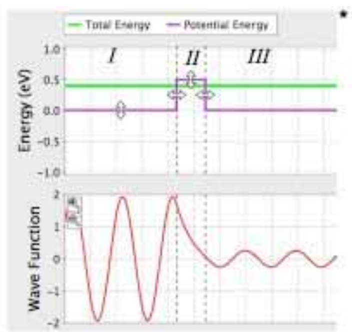
Solutions relative to A, F must be 0.

$$R = \frac{|B|^2}{|A|^2} \qquad T = \frac{|E|^2}{|A|^2}$$

Structure of coefficients depends on understanding of particle current.

Novice vs. Expert

- The novice solution gets the math



$$\psi_I = Ae^{ikx} + Be^{-ikx} \qquad k^2 = \frac{2mE}{\hbar^2}$$
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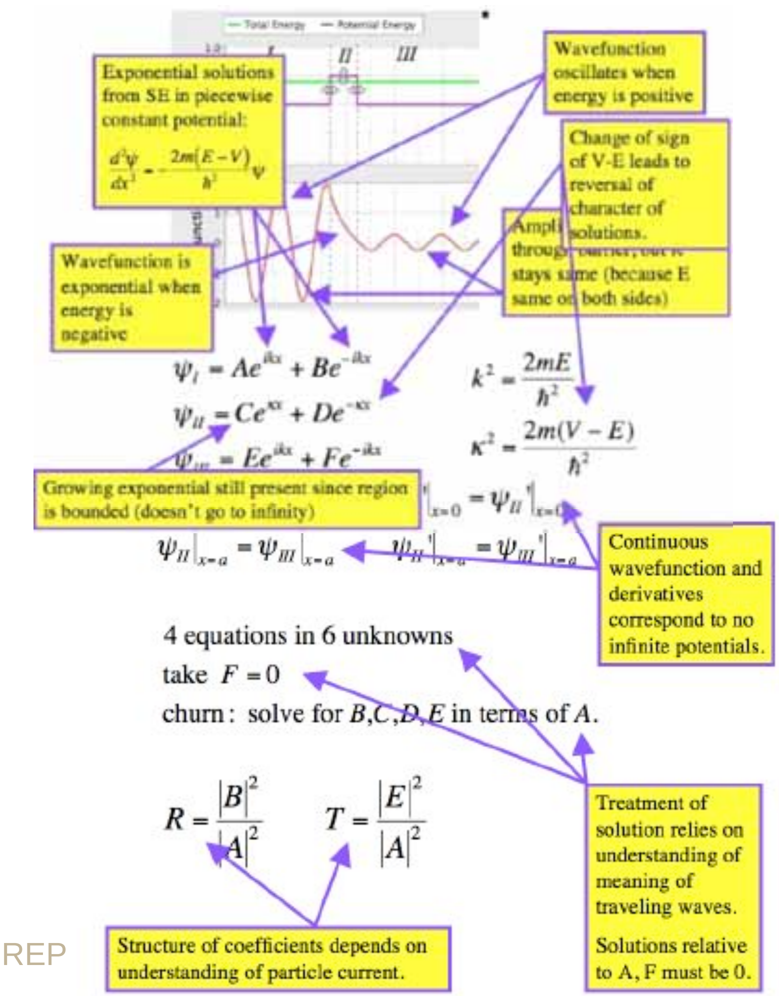
4 equations in 6 unknowns
take $F = 0$
churn: solve for B, C, D, E in terms of A.

$$R = \frac{|B|^2}{|A|^2} \qquad T = \frac{|E|^2}{|A|^2}$$

- The expert adds a web of physics associations



Physical Meaning

- In both the earlier examples and in this QM example, the physical interpretation of the math provides a richer structure – an "encyclopedic" or "contextual" interpretation.
- In the examples of the electric field and photoeffect, the physical associations changed the way that the math was interpreted.
- In the QM example, the physical associations provided a stability allowing us to detect and correct errors.

Implications

- In teaching mathematical topics in advanced physics, we need to “unpack” the math and the physics and find ways to help students put the two together.
- We need both research on the paths students take to blend math and physical ideas and instructional environments that emphasize the connections.

Conclusion

- "Mathematical semantics" plays a critical if underappreciated role in physics instruction.
- The language and tools developed by cognitive linguists to make sense of sensemaking in everyday language may turn out to be very useful for us.