Students' Use of Mathematics in the Context of Physics Problem Solving: A Cognitive Model

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Although much is known about the difference between expert and novice problem solvers, knowledge of those differences typically does not provide enough detail to help instructors understand why some students seem to learn while solving problems and others do not. A critical issue appears to be how students use the knowledge they have in the context of solving a particular problem. In this paper we outline a theoretical cognitive model for making sense of how students use mathematics in the context of physics problems. The model is developed within the theoretical framework of resources. We identify four classes of fundamental mathematical resources and six organizational structures or epistemic games. Each game is a locally coherent associational pattern of control structures (expectations) activating resources and processes (moves) within the specific example. The hypothesis that students tend to function within the narrow confines of a fairly limited set of games provides a good description of most of our data. We demonstrate the use of these resources and games in examples taken from videoclips of students solving problems in a two-semester reformed algebra-based physics course at the University of Maryland. Implications for instruction are discussed.

KEYWORDS: problem solving, epistemic games, expectations, resource model, cognitive modeling

1. Introduction

Students learning physics at the college level often have considerable difficulty with the mathematical problem solving that is an integral part of most physics classes. Instructors may assume that these difficulties arise from a lack of mathematical skills, but little evidence has been presented to determine whether or not this is the case. As part of a project to reform introductory algebra-based physics,[2] we have collected extensive data of students learning physics and solving physics problems in a variety of classroom environments. This data includes some remarkable student behavior, such as

- students failing to apply the same reasoning with complicated numbers that they can easily do with small numbers;
- students rejecting their correct mathematical reasoning in favor of incorrect intuitive reasoning;
- students using incorrect qualitative (p-prim) arguments to rebut a qualitative argument even when they know the correct formal argument;
- students failing to use personal knowledge they know very well in favor of misinterpretations of authority-based knowledge when reasoning in a formal context.

These behaviors are often quite robust, with students dramatically ignoring — appearing not even to hear — explicit suggestions from an instructor. As a result, these behaviors look like what one might crudely describe as "misconceptions of expectations" about to how to solve problems.

In order to make sense of this data, we construct a theoretical model that allows us to describe the cognitive processes that students use – correct and incorrect – in the context of applying mathematics in physics. Building on and extending ideas developed by diSessa and Sherin,[3,4,5,6] and by Collins and Ferguson,[7] we identify cognitive tools that students bring to bear on mathematical problem-solving tasks in the context of physics. Our theoretical model fits into the more general theoretical superstructure we refer to as *the resource model*.[8,9,10,11] In this broad model of student thinking, knowledge elements combine dynamically in associative structures activated by control structures in response to inputs from each other and from the environment.

Our theoretical framework offers researchers and educators a vocabulary (an ontological classification of cognitive structures) and grammar (a description of the relationship between the cognitive structures) to describe students' understanding and use of mathematics in the context of physics.[12] Viewing student activity through the lens created by this framework can help researchers and educators (i) make sense of the dramatic context dependence in students' use of mathematics in the context of physics and (ii) understand how teacher-student interactions can more effectively help students develop their own problem solving skills.

In the next section, we give a brief overview of our theoretical framework, that is, a structure of assumptions and mechanisms that provide a framework for the construction of specific cognitive models of student thinking. In section three, we describe the setting of the study: the student population, the modified instructional environment, and the methodology used to collect and analyze our data. In section four, we present a cognitive model for mathematical problem solving in physics: mathematical resources and epistemic games. In section five, we use our theoretical model to analyze a one-hour student problem-solving session. In the final section, we discuss some instructional implications, and present some conclusions. Much of the work described here is taken from the dissertation of Jonathan Tuminaro and more detail can be found there.[13]

2. The Theoretical Framework

Constructivism — the idea that a student constructs new knowledge based largely on what that student already knows — is the dominant paradigm in modern educational theories. The teacher's role in the constructivist paradigm is to create environments that help students undertake this construction accurately and effectively. In order to do this, it helps the teacher to know (i) the content and structure of the students' existing knowledge and (ii) how the students use this knowledge to construct new knowledge. There has been considerable direct observational research on the difficulties students have with various items of physics content,[14,15] but to understand how students organize, access, and use their existing knowledge requires a finer-grained understanding of how students think and respond. We need some theoretical understanding of the basic elements of fundamental cognitive activities and how they are organized.

Previous Research

Research on students' naïve knowledge and on expert/novice differences in problem-solving are two topics that are particularly relevant to the current study. In this subsection, we give a brief review of these two areas of research.

Students' Naïve Knowledge

The fact that students bring prior naïve knowledge into a physics class has been well documented in the research literature.[14,15] The level of abstraction at which the students' naïve knowledge is described, however, varies considerably. Some researchers describe student knowledge that does not align well with the scientific knowledge we are trying to teach as "misconceptions," "alternative conceptions," or "naïve theories." These researchers assume that students have internally consistent models of how aspects of the world work.[16,17,18,19] Others describe the knowledge of beginning students in physics as fragmented and spontaneous.[3,20,]

Expert/Novice Differences in Problem-Solving

Researchers have studied problem-solving in different contexts: problem solving associated with games such as chess,[21] problem solving in mathematics,[22,23,24] and mathematical problem solving in the context of physics.[25,26,27,28,29,30,31] There is agreement that there are substantial differences between experts and novices; experts have more knowledge and organize it better. But most attempts to model the differences at a finer scale have focused on creating computer models that would solve problems effectively. Sometimes these models are algorithmic;[32] sometimes they are based on heuristics extracted from expert informants.[33] While these approaches can produce computer software that can carry out some tasks that human experts do, it is not at all clear that they correctly model how a human being learns and functions. (A good summary of the successes and limitations of this approach is given in d'Andrade.[34]) Nor do they help us understand how to help students make the transition from novice to expert.

If our goal is to teach a human being effectively, it is appropriate to build a theoretical model based on our knowledge of the functioning of that system and not some other. The resource model is based on a combination of three kinds of knowledge about the functioning human: from neuroscience, cognitive science, and behavioral science. It permits us to begin to create a finer-grained understanding of student behavior that can bridge the alternative and fragmented conception models and can help us develop a more detailed understanding of the novice-to-expert transition.

Researchers in neuroscience, cognitive science, and behavioral science attempt to model human thought at a variety of grain sizes. The resource model attempts to build a synthesis of principles, extracted from neuro-, cognitive, and behavioral science, that offer a mechanism that can help us understand student thinking. Much has been learned, though one has to be cautious in applying research results at a fine-grained level from neuroscience or cognitive science to real-world situations. Nevertheless, results from neuroscience give some guidance as to the kind of structures that might be relevant for the coarser-grained models needed to deal with real students in real situations. Note that in constructing this synthesis we are not attempting to create a fundamental theory of human behavior. Rather, we are developing a theoretical framework or superstructure[9] within which plausible phenomenological models can be created that can help us understand what we see in our classroom, but that are also consistent with what is known about the fundamental mechanisms and operation of the brain.

The Neural Basis of Cognition

A model of cognition that is consistent with and is supported by results from neuroscience and cognitive science is synthesized and documented in many books.[35,36,37,38] In this model, cognitive elements of knowledge and memory are represented by networks of connected neurons. When someone recalls or uses the knowledge represented by a particular network, the neurons of the network are *activated* (increase their firing rate).[39] Particular knowledge elements tend to be multimodal (i.e., to involve activation and interpretation of multiple sensory and interpretive structures) and involve neurons in many parts of the brain.[40] Cognitive networks arise from the building of associations among neurons through synapse growth.[41] The association of neurons can vary in strength and increases with repeated associational activations.[42]

Neural connections can be excitatory or inhibitory.[41] This creates the possibility of *executive processes* that result in the selective activation of some networks and the suppression of others.[43] fMRI studies and neurophysiological studies with patients who have brain lesions suggest that the pre-frontal cortex is a primary site of a large number of control structures (though they are expected to occur in other parts of the brain as well).[44,45,46]

A Model of Cognition for Education: Resources

We work within the framework described that attempts to create a model that is consistent with the neuro-cognitive model described above: the *resource model*.[9,10,47] We want the model to be sufficiently coarse-grained that it allows us to describe observed student behavior and sufficiently fine-grained that it gives us insight into the mechanisms responsible for those behaviors. The critical elements of the model are the basic knowledge elements, the way those elements are linked, and the way those linked structures are activated in different circumstances. *Knowledge elements* include both declarative and procedural knowledge. We refer to the linking patterns of association as *knowledge structures* and to the executive function that determines when those structures are activated as *control structures*. We broadly refer to all the elements of this model — basic knowledge elements, associational patterns, and control structures available to students thinking about a physics problem — as *resources.*

Basic Knowledge Elements: Compilation

A network corresponding to an element of knowledge becomes robust through practice and experience. For example, one can quickly and easily identify the combination of sensations associated with holding a cup of hot coffee. We effortlessly combine the perception of the pixels (activation of rods and cones) on our retina with the touch, smell, and taste of the coffee into a perception of what appears to be a single object. Neuroscientists call this *binding*, but we prefer to say that it is *compiled*.[48] Compiled knowledge structures are seen as irreducible by the individual and can be used as a single chunk in working memory[49] and are referred to as knowledge elements.

Note that a knowledge element may have a structure and that for some purposes it might be useful to decompose them into finer-grained knowledge elements even when the user sees them as irreducible. This is like considering molecules consisting of atoms. For some tightly bound molecules in some situations (e.g., molecules in a gas in kinetic theory) it suffices to consider the molecule as a single functioning unit without substructure. In other circumstances (e.g., situations in which chemical reactions occur) or for more weakly bound molecules, it is essential to keep the molecule's structure in terms of atoms in mind.

Our model has knowledge elements of four types: intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices. These are discussed in detail in section 4.

Patterns of Association: Knowledge Structures

Because cognitive networks are extended and because neurons have large numbers of synapses with other neurons, an individual neuron may be a part of multiple mutually linked knowledge structures. As a result, activation of one network may result in the associated activation of other networks. Patterns of association develop, linking different resources in different situations. The patterns of association individuals develop may help or hinder them in solving physics problems.[11] Learning occurs as the result of the growth of new synapses that result in changing the topology of existing networks.[50]

Executive Function: Control Structures

Neural executive processes make possible extensive structures of selection and control that occur at all levels of neural networking from the shutting off of a reflex arc[41] to conscious decision making.[24] A classic example of a control structure is exhibited in the Stroop task.[51] In this classic cognitive experiment, subjects are presented with a series of words printed in different color inks and asked to name the colors of the ink for each word. For example, if the word "house" is printed in blue ink, the subject has to say "blue." The time is recorded for the subject to read the colors of 25 words. The task is then repeated, but in the second pass, the words are themselves color words. Thus, the word "red" might be printed in blue ink and the subject has to say "blue." When the words and the colors conflict, the task is much more difficult and the time to complete it increases dramatically. The explanation is that the subject is receiving two different kinds of conflicting color cues: the meaning of the word and the observed color of the ink. These twp pieces of information enter the brain in distinct places and have to be reconciled with a decision as to which color signal to select before making a motor response (speech). An fMRI study of a subject performing the conflicting-information part of Stroop task shows strong activation of sites in the pre-frontal cortex, in contrast with what happens in the non-conflicting information part of the task.[52]

Control structures play a powerful role in implementing students' expectations of what resources and knowledge structures are appropriate to activate in particular circumstances. They may be organized a variety of ways. Specific knowledge resources may be organized into various locally coherent patterns of association and these patterns have implications for student responses. For example, a student given a physics problem may activate related information about force and motion or, alternatively, about work and energy. The student may be able to use each of these sets of information coherently but not put them together in an integrated fashion.[11] Another proposed organization of knowledge structures is the coordination class.[5]

In this paper, as part of our model of students' mathematics use in physics, we propose that a useful way to analyze the control structures of expectations is to describe them in terms of locally coherent, goal-oriented activities. We choose to call these epistemic games because of their similarity to the structures proposed by Collins and Ferguson (C&F).[7] Note that the activities described by C&F were normative — activities carried out by experts to solve problems. We extend the idea to one that is descriptive of observed student behavior. (Note that some other researchers have also extended their use of the term in this way.[53,54])

A critical component in the construction of the coarse-grained theory is the detailed observation of student behavior in real-world situations. In this paper, we report on a research study of authentic in-class (ecological) student behavior. These observations allow us to identify knowledge elements, knowledge structures, and control structures that have considerable power to explain how students in introductory physics classes use mathematics to solve physics problems. How the control structures respond to the socio-cultural environment is also of considerable importance and interest. It will be discussed in another paper.[55]

3. The setting of the study

This study was done as a part of a research study carried out at the University of Maryland[2] to determine whether an introductory physics course could serve as a venue to help biology students learn to see science as a coherent process and way of thinking, rather than as a collection of independent facts; and whether this goal could be achieved within the context of a traditional large-lecture class without a substantial increase in instructional resources. The project adopted reforms that were well-documented to produce conceptual gains and adapted them to try to create a coherent package that produced epistemological and metacognitive gains. The hope was that this could be done without sacrificing the conceptual gains associated with these reforms. (This turns out to be the case with epistemological state measured by pre-post MPEX and conceptual state measured by fractional gains on the FCI. Strong gains were obtained in both measures. These results will be documented elsewhere.)

Data on the student responses to the modified environment were collected in a variety of ways in order to provide triangulation on the learning process of individual students and evaluations of the overall class results. The learning environments were constructed to encourage students to learn in group discussions taking place both in and out of the classroom context. Hundreds of hours of these group discussions were recorded by video cameras and provide the bulk of the data for this study. In addition, all student homework, quizzes, and exams were scanned before grading. Finally, we gave pre-post conceptual (FCI[56] and FMCE[57]) and epistemological attitude surveys (modified MPEX[58]).

Student Population

The students in this study were enrolled in an introductory, algebra-based physics course. They were approximately 60% female; more than 70% were juniors and seniors, about 50% were biological science majors, and about 40% were pre-meds. (There was some year-to-year fluctuations in these numbers.) A particularly interesting statistic for this study is that more than 95% of the students had successfully completed two semesters of calculus, yet they chose to enroll in an algebra-based introductory physics course despite the availability of a calculus-based alternative. Data were collected in 10 semester-long classes over a four-year period from a total of more than 1000 students.

Structure of the modified course

The introductory, algebra-based physics course at the University of Maryland (UMd) was reformed by the Physics Education Research Group (PERG) as part of the research study. The course had four major structural components. The homework, the lecture, the discussion, and the laboratory were all modified to be non-traditional in some fashion. In addition, we attempted to make all parts of the course coherent with each other. We believe that the overall epistemological orientation of the class was responsible, at least in part, for the students' willingness to spend long times working together on individual problems and encouraged some of the behaviors we observed. Therefore, we discuss our reforms in some detail in the following sections.

Homework problems: Homework problems are particularly important for this study, since the observations reported here are videotaped sessions of students working on homework. Problems were regularly assigned and graded. The problems assigned were not traditional end-of-chapter textbook exercises. Instead, they included a mix of challenging activities including representation translation problems, context-based reasoning problems, ranking tasks, estimation problems, and essay questions with epistemological content. (For more on these types of problems see chapter 4 of Redish's *Teaching Physics*.[59])

The instructor expected that each problem would take the students about an hour to complete, and he communicated this expectation to the class. In accordance with his expectation, the instructor only assigned about five problems each week. (The specific problems we discuss here are given in the Appendix.) Because these problems were assigned as homework and graded, our observations of students working on these problems gave us an authentic look at how students actually behave in real-world classroom problem-solving situations – as opposed to watching them solve problems artificially posed to them in an interview environment.

Lecture: The lecture was given by the instructor (Redish) in a large lecture hall to about 100-160 students. The class met three times a week for fifty minutes. Two modifications to the lectures significantly increased student attendance and participation.

(1) The *Peer Instruction* environment[60] was adapted for this class. Each student was issued a *remote answering device* (RAD). The instructor periodically asked a multiple-choice question during the lecture to which the students responded using these devices. A computer automatically displayed a histogram of the results. This reform was modified to focus on epistemological issues. Discussion before the question often focused on students' intuitions based on their real-world experience. Discussion after the question often focused on the wrong answers, why they were chosen, and whether even they had a "correct" intuitive core. The goal was to encourage students to not just "know" the right answers, but to perceive them as both plausible and intuitive.

(2) The *Interactive Lecture Demonstration* (ILD) environment[61] was adapted for this class. About a half-dozen times per semester, students received a worksheet outlining specific questions that were to be discussed. The instructor led the students through the worksheet and led a class discussion about the issues raised in the worksheet. We modified the standard Sokoloff-Thornton procedure so that there was only a single worksheet that emphasized finding the valid content of a student's intuition and refining it.

The students were not graded on their answers to either the RAD questions or the ILDs, but they were given participation points for doing them and were given homework and test questions to assess their understanding of the material discussed during ILDs.

Discussion and Laboratory: Each week students attended a discussion and laboratory section taught by a teaching assistant. These sections were limited to 20 students per section and met once a week for three hours. In the first hour the students had a discussion session and worked in groups of four on tutorial worksheets. Some of the worksheets were adapted from the tutorial environment developed at the University of Washington[62] to be more epistemologically explicit, while others were adapted from worksheets previously created at the University of Maryland.[63]

During the second and third hours, the students worked on a reformed laboratory environment called *Scientific Community Labs*.[64] These laboratories were non- traditional. First, instead of a lab manual with detailed instructions, students received a brief description of a particular setting and were asked a question whose answer was to be determined experimentally. Working in groups of four, they were expected to design and carry out an experiment to answer the question. Second, the laboratories focused on the process of doing science, rather than on physics content. Topics were chosen whose answers were "not in the book" or were covered much later in the course. The goal was to foster experimental exploration and discussions of "how do you know and why do you believe your results?"

Coherence: An important characteristic of the reformed class was the attempt to make the various parts epistemologically oriented and mutually supportive. The instructor and the teaching assistants frequently cross-referenced among homework, lectures, tutorials, and laboratories. Exam questions drew from and mixed information that the students had worked on in each of the class components.

The Course Center

Since the traditional discussion sections were converted to tutorials, the students did not have time to discuss the problems on the homework set with a TA during these periods. To close this gap, a room was set up, called the *course center*, where students could gather to work on the homework problems together. Most of the data for this study comes from videotaped sessions of students working on homework problems in the course center.

A teaching assistant or instructor was available in the course center approximately twenty hours per week. The TA or instructor was present to offer assistance but not to explicitly solve the problems for the students, as they often do in many traditional recitation sessions. The special features of this room were its architecture, the white boards, and the audio-video set up.

Architecture: Many students expect recitation sessions in which a teaching assistant stands at the front of the room and solves problems, while the students frantically copy down the solutions. It's another lecture, but one that models problem solving. The architecture of the course center was altered to modify this expectation by removing the front of the room. All the chairs with desk arms were removed, and they were replaced with five long workbenches and stools. (See figure 1 for a schematic lay out.) This seating arrangement did not direct the attention of the students to any one location in the room – as is the case in all lecture halls in which the seating is arranged to face the 'front,' directing attention to the lecturer. The natural focus of attention of a student seated at one of these worktables is the work area in front of them and the student seated across from them.

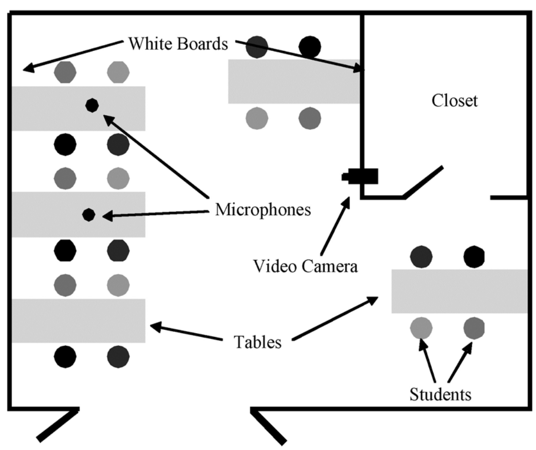

Fig. 1: Top-view of the lay-out of the course center.

Whiteboards: As a second alteration to the course center, whiteboards were mounted on the walls and the students were provided with dry erase markers. The reason for this was threefold. First, the location of the whiteboards made them difficult to reach for the TAs but easy for the students – an architectural feature that encourages the students to go to the whiteboards and discourages the TAs from solving the problem for the student or "lecturing" at the whiteboards. Second, the whiteboards facilitate group problem solving. Research on expert and novice problem solving has shown that external representations are a helpful and sometimes necessary tool in the problem solving process.[23,30] The whiteboards offered the students a medium to share their external representations with each other. Third, the whiteboards helped with our research agenda. The students' shared representations on the whiteboards are visible to the video camera.

Audio-video set-up: The course center was equipped with a digital video camera and microphones were mounted in the middle of the tables to ensure quality audio reception. The video camera was mounted about seven feet above the floor on the wall of the closet across from the tables that were equipped with microphones. The elevation of the camera meant that students and staff members walking by the closet would not be in the camera's field of vision, that students sitting closer to the camera did not block the students who sat closer to the wall, and it allowed a clear view of what the students wrote on the whiteboards.

Methodology

The data set. The majority of the data for this study comes from about 60 hours of video-taped sessions of groups of students solving homework problems in the course center. Additional videotaped data comes from a tutorial session involving a discussion about conservation of momentum. In addition to the video data, all the homework and exams that the students turned in were electronically scanned and stored on compact disc. The scanned homework data served to corroborate video data collected in the course center.

Selecting episodes. Sixty hours of video is too much to be analyzed in detail, so the full data set was reduced by using the following selection criteria:

1. *Episodes rich in student thinking*. Since this is an ecological study, the students themselves determine how they choose to work on the problems. Students solving problems in a group may go several minutes without speaking. Any video containing discussions rich in articulated student thinking was flagged for further investigation.
2. *Episodes rich in mathematics use.* Since this is a study about the use of mathematics in physics, the selection above was further refined to focus on segments that contain students using and discussing mathematics.

These selection criteria reduced the full data set to about 11 hours of video that were analyzed in detail.

Transcribing and coding the episodes. These 11 hours of video were transcribed and analyzed to identify the resources that students activated to solve the problem they were working on. Working within the theoretical frame described above, we identified several recurring patterns of student behavior. In order to describe these observations, we created a synthesis bringing together and extending theoretical work on the use of mathematics.

4. Mathematical Resources: How students understand mathematics in physics

Students have a wealth of previous knowledge and ideas that they bring to bear when using mathematics in the context of physics. In order to understand and talk about what students are doing, we need to have a description of the students' resouces and the way they organize them. In this section, we identify four different kinds of relevant knowledge and six locally-coherent organizational control structures (epistemic games) that we have seen students use.

Mathematical Knowledge Structures

The mathematics education literature contains many discussions of the cognitive processes involved in doing math or physics. We select here four broad general resources that were frequently activated in our observations: intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices.

Intuitive Mathematics Knowledge

The mathematics used in physics is a formal, rigorous subject matter that takes years of schooling and practice to learn; however, many of the cognitive building blocks necessary to understand this subject are present in very young children – even infants. We refer to these fundamental cognitive building blocks as *intuitive mathematics knowledge*. Research involving human infants demonstrates their ability to differentiate sets of one, two, and three objects from each other.[65] This ability has been dubbed *subitizing* and has also been observed in various species of primates and birds. Another, more familiar, cognitive building block that is necessary to understand mathematics in physics is *counting* – a cognitive ability that should be familiar to all readers.

The concepts of subitizing and counting are particularly important for understanding students' use and understanding of mathematics in physics at the introductory college level. We examine the episode of Mary[66] discussing her approach to part (c) of the Conversion Problem (given in the Appendix, problem #1):

Mary has difficulty, so she calls the TA over:

> *…alright, if I convert 65 mph to feet per second, which is the other thing that's given in feet… So then I got 95 feet per second is what you're moving, so in 500 feet like how long? So, I was trying to do a proportion, but that doesn't work. I was like 95 feet per second...oh wait...yeah in 500 feet, like, x would be like the time...that doesn't—I get like this huge number and that doesn't make any sense.*

Mary correctly identifies that using a proportion could help her solve this problem, but has trouble implementing this strategy. It appears that she has created the proportion: $\frac{x}{500} = \frac{95\text{ feet}}{1\text{ second}}$. When she cross-multiplies she gets a "huge number" that "doesn't make any sense."

The TA attempts to redirect Mary:

> *So what if I said something like...if you're traveling 8 feet per second and you go 16 feet, how long would that take you?*

The TA changes how Mary approaches this problem by replacing 95 feet per second and 500 feet with 8 feet per second and 16 feet, respectively. With this replacement, Mary immediately responds "2 seconds." Her immediate response is an indication that the knowledge she uses to arrive at this answer is readily available to her, suggesting she is using intuitive mathematics knowledge. In particular, she could be *counting* or *subitizing*. That is, she could be counting up the number of seconds needed to make up 16 feet. Alternatively, she could be visualizing the number of '8 feet per second' blocks in '16 feet,' then using her subitizing ability she arrives at the answer of 2 seconds.

The evidence in this case does not distinguish between these interpretations. However, the evidence does indicate that changing the numbers in the problem cues Mary to activate a new set of resources: intuitive mathematics knowledge, usable in a direct fashion without using any specific chain of reasoning. (This is most likely, given the speed of her response.) In Mary's initial approach she attempts to use a formal, symbolic approach involving proportions, but does not set her proportion up in a way that matches the physics of the problem. By using "easier numbers," Mary is able to tap into intuitive knowledge that she already has to construct a general relationship between distance, speed, and time – a relationship she then uses to get the answer to the problem as it was originally stated.

Intuitive Mathematics Resources	
Subitizing	The ability to distinguish between sets of one, two, and three objects.
Counting	The ability to enumerate a series of objects.
Pairing	The ability to group two objects for collective consideration.
Ordering	The ability to rank relative magnitudes of mathematical objects.

Table 1: Intuitive mathematics resources

This episode illustrates that the use of intuitive mathematics knowledge can serve as a connection for students to the more sophisticated and formal mathematics used in college level physics. We do not offer an exhaustive list of intuitive mathematics knowledge. We are simply drawing attention to the fact that instructors can use this aspect of students' previous knowledge during formal instruction that involves more advanced mathematics. Lakoff and Núñez[67] offer a more extensive list of primitive cognitive capacities, including ordering and pairing, that are required for and involved in advanced and abstract mathematical thought. Table 1 lists some different forms of intuitive mathematics knowledge they identify.

Reasoning Primitives

In addition to intuitive mathematics knowledge, students use a form of intuitive knowledge about physical phenomena and processes that they have learned in their everyday life experiences to make sense of the physical world. DiSessa[3] proposes that students develop an intuitive sense of physical mechanism from abstractions of everyday experience. This intuitive sense of physical mechanism arises from the activation and interaction of multiple cognitive resources that he refers to as *phenomenological primitives* (p-prims).

The name, *phenomenological primitives*, is used to convey several key aspects of these cognitive structures. The word "phenomenological" is used to reflect the idea that these resources are abstracted from everyday phenomena. (*Closer is stronger* could be abstracted from the phenomena that the closer one is to a fire the warmer it is.) These resources are "primitive" in the sense that they are "irreducible and undetectable" to the user – they are often used as if they were self-explanatory. (Asked why it is warmer closer to a fire, a student using *closer is stronger* may respond, "it just is."[68])

Because of his focus on the irreducibility of p-prims with respect to the user, diSessa identifies p-prims at differing levels of abstraction: for example, *force as mover* and *abstract balancing*. *Force as mover* involves the very specific concept of an object moving under the influence of a force; whereas, *abstract balancing* involves the very general notion that two unspecified influences can be in a state of equilibrium. Because of the specific nature of p-prims like *force as mover*, diSessa proposes that there are thousands of p-prims corresponding to the myriad of physical experiences one may have in this complex world.

To reduce the extremely large number of p-prims and discuss cognitive structures that exist at the same level of abstraction, we follow Redish[9] and abstract from p-prims the notion of intuitive pieces of knowledge called *reasoning primitives*. Reasoning primitives are abstractions of everyday experiences that involve generalizations of classes of objects and influences. In this view a p-prim like *force as mover* results from mapping an abstract reasoning primitive like *agent causes effect* into a specific situation that involves forces and motion. We refer to a reasoning primitive that is mapped into a specific situation as a *facet*. The specific agent, in this case, is a force and the effect it causes is movement. *Agent causes effect* could also be mapped into *force as spinner*, another p-prim identified by diSessa.[3] This makes it clear how the notion of reasoning primitives reduces the total number of resources necessary to describe students' previous knowledge about physical phenomena (compared to p-prims). In addition, *agent causes effect* and *abstract balancing* both reflect relationships between abstract influences, and therefore exist at the same level of abstraction.

Another reason to consider the reasoning primitives underlying facets is to understand process components that may be addressable by instruction. If a student is using an appropriate reasoning primitive but has mapped it inappropriately it may be simple to help the student change the mapping. This more fine-grained theoretical model activates different instructional responses than if one considers a particular p-prim to be an irreducible and robust "alternate conception."

Symbolic Forms

In the previous section, we saw how students can use an intuitive sense of physical mechanism to understand various physical situations. Sherin[4,6] considers the cognitive mechanisms and processes involved when students look at an equation and interpret its meaning. He argues that students use an intuitive sense of physical mechanism in concert with knowledge of mathematical symbolism and protocols to make sense of equations in physics. In order to understand and describe how students use and understand physics equations, we need two cognitive constructs: a symbol template and conceptual knowledge.

The *symbol template* is an element of knowledge that gives structure to mathematical expressions; e.g. □ = □ or □ + □ + □... (where the boxes can contain any type of mathematical expression). That is, the symbol template is a general symbolic relationship pattern into which specific quantities can be mapped. The *conceptual knowledge* is a knowledge structure that offers a conceptualization of the knowledge contained in the mathematical expression. The conceptual knowledge is typically, for the simple examples of equations considered in algebra-based physics, analogous to diSessa's p-prims. It is a direct mapping of an interpretive meaning onto a symbolic structure. (For more complex equations, more sophisticated knowledge structures may be required.) A *symbolic form* is the combination of a symbol template and conceptual knowledge.

An example of a student deriving an equation for air drag in the Air Drag Problem (Appendix, problem #2) facilitates this discussion about symbolic forms.

Amy: *So basically what you have to do-*

Monica: *So like when you think about it, you can think that if you increase density, the air can - that - it would have to be directly proportional, cause you increase density, the resistance with the air has to also increase.*

Amy: *Yeah. So...*

Competing Terms Cluster		Terms are Amounts Cluster	
Competing Terms	□ ± □ ± □ …	Parts-of-a-Whole	[□ + □ + □ …]
Opposition	□ - □	Base ± Change	[□ ± Δ]
Balancing[*]	□ = □	Whole – Part	[□ - □]
Canceling[*]	□ - □ = 0	Same Amount	□ = □
Dependence Cluster		**Coefficient Cluster**	
Dependence	[…x…]	Coefficient	[x □]
No Dependence	[…]	Scaling	[n □]
Sole Dependence	[…x…]	**Other**	
Multiplication Cluster		Identity	$x = \ldots$
Intensive•Extensive	$x \times y$	Dying Away	$\left[e^{-x\ldots}\right]$
Extensive•Extensive	$x \times y$		
Proportionality Cluster			
Prop+[◊]	$\left[\frac{\ldots x \ldots}{\ldots}\right]$	Ratio	$\left[\frac{x}{y}\right]$
Prop-	$\left[\frac{\ldots}{\ldots x \ldots}\right]$	Canceling(B)	$\left[\frac{\ldots x \ldots}{\ldots x \ldots}\right]$

Table 2. Symbolic forms identified by Sherin (refs. 4 and 6)

Monica: *And as you increase the radius, that also increases. So they're all directly proportional-*

Amy: *Right*

Monica: *So you multiply them-*

Amy: *Right, so it's all multiplied-*

Monica: *Instead of dividing them.*

Monica has the reasoning primitive *more is more* activated when she states that "if you increase density…the resistance with the air has to also increase"; *i.e.* more density is more resistance. This conceptual idea is associated with the symbol template □ = […x…]. The left side of the equation is associated with the drag force. The density appears on the right side of the equation since it is directly proportional to the drag force it. Therefore, the drag force (D) and density (ρ) are mapped into the symbol template, □ = […x…], resulting in the specific expression, D = […ρ…]. Monica goes on to identify that an increase in radius also results in an increase in air drag, which is also associated with the symbol template □ = […x…], *i.e.* D = […□…]. Since an increase in density and radius both result in an increase in resistance, Monica realizes that they both must appear in the numerator: "So you multiply them." The association of the conceptual schema of *more is more* with the symbol template □ = […x…] occurs often in students' interpretive utterances, and is given the name *proportionality plus* (*prop+*, for short).

Sherin identifies collections of symbolic forms, which he organizes into clusters. The symbolic forms within a given cluster tend to involve "entities of the same or similar ontological type. For example, "[symbolic] forms in the Competing Terms Cluster are primarily concerned with influences."[69] That is, symbolic forms in the Competing Terms Cluster do not involve specific physics concepts (like force or velocity), rather they involve everyday concepts (like push or motion). Table 2 lists the different clusters and symbolic forms that Sherin identifies. We draw out examples of *balancing* and *canceling* from our data set and discuss them below.

Interpretive Devices.

Symbolic forms cannot be the entire story for how students understand and interpret equations. Students (and experts) appear to have compiled strategies for extracting information from physics equations. We follow Sherin and call these compiled strategies interpretive devices.[70] Sherin identifies three different classes of interpretive devices – Narrative, Static, and Specific Moment – that students in his data corpus use to interpret physics equations. In addition to these three, we propose a fourth class of interpretive devices: intuitive interpretive devices. Table 3 lists the different interpretive devices according to class.) The interpretive devices in the Narrative, Static, and Specific Moment classes all derive from and rely on the formal properties of equations. Therefore, we will lump all of these classes into one class, which we call formal interpretive devices. In contrast, intuitive interpretive devices are reasoning strategies that are abstracted from everyday reasoning and applied to physics equations.

The students' first attempt seems correct:

Narrative	*Static*
Changing Parameters[*] Physical Change Changing Situation	Specific Moment Generic Moment Steady State
Special Case	Static Forces
Restricted Value Specific Value	Conservation Accounting
Limiting Case	*Intuitive*[a]
Relative Values	Feature Analysis[*] Ignoring[*]

Table 3. Interpretive devices by class

Arielle*: So then the F_{net} for A, the F_{net} for M. This is a big mass and this is a little mass and [the Δt]*

[*] Discussed below.
[a] Class of interpretive devices not identified by Sherin.

are equal, so this has got to be a big, what is it, a big velocity and this has got to be a small velocity. So, p for A and p for M – the change in velocity here has got to be sort of bigger. Big velocity little mass. Big mass little velocity. But [the net forces] are equal.

Tommy*: Right.*

Betty*: Right.*

Arielle: *So the momentums got to be the same, right?*

It seems that Arielle is using *prop+*: the mass and the velocity are directly proportional to the net force. In addition, it appears that she is using a particular strategy for extracting meaning from this equation – in this case, the formal interpretive device called *changing parameters*. *Changing parameters* is an interpretive device in which "a quantity, usually corresponding to an individual symbol in the expression, is imagined to vary while other quantities are held fixed."[71] Arielle imagines how changing a parameter on the right side of the equation (*i.e.* mass and change in velocity) will affect quantities on the left (*i.e.* the net forces). Since glider A has a smaller mass than glider M she imagines changing the values of the change in the velocities to maintain the equality between the forces. Figure 2 shows this reasoning schematically.)

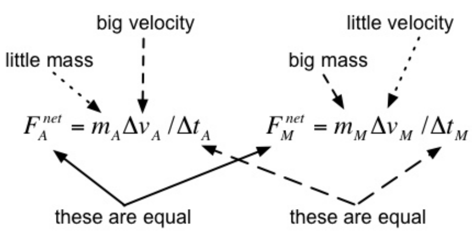

Fig. 2: Interpretation of equations using the formal interpretive device changing parameters*.*

At first glance Arielle's reasoning appears to be very good. However, she is not satisfied with the conclusion that the momenta should be the same, so she continues the discussion:

Arielle: *I don't know… No, this is not right.*

Betty: *It's right. But—I think it's right, but it's like--*

Tommy: *No, I think that's correct.*

Betty: *...but see you have the subset so you have the change—the change in momentum...*

Arielle: *But the change in velocities are not the same though.*

Betty: *The change—*

Tommy: *Yeah, the change in velocities aren't the same. And also—*

Arielle: *Yeah, that's the problem, I was thinking they were the same.*

The first line in this set of quotations indicates that Arielle is uncertain about the conclusion that the momenta would be the same. However, at first glance it appears that the last line in this set of quotations is in direct contradiction with what Arielle had said in the first set of quotations. In the first set of quotations she had said that the change in velocity for glider A had to be large, while the change in velocity for glider M had to be small; now, however she's stating that she was thinking the change in velocities were the same. This seems like a contradiction; however, what she says later helps clear up this apparent contradiction.

Tommy: *Momentum might—could be the same. It could be.*

Arielle: *All right...they're in opposite directions.*

Tommy: *Wait, wait, wait. They're in opposite directions but they could be the same.*

Arielle: *Opposite directions—how could they be the same? If the masses are different and the change in velocities are different, the momentums can't be the same.*

It appears that Arielle is using a different interpretive device than she was before to conclude that the momenta cannot be the same. We suggest that she's using the intuitive interpretive device of *feature analysis* – a form of pattern recognition in which the features of a stimulus are evaluated individually. For example, in deciding whether drawings of two faces, observed at two different times or in two different places, represent the same or different individuals, one might run through a variety of comparisons to decide. The more differences that are found, the more likely it is that the objects compared are different.

In Arielle's analysis of the momenta, she is comparing the features of the individual momenta (the features of the momenta are the masses and change in velocities). The more features that are different between the two momenta the easier it is to tell that the two momenta are different. (See figure 3 for a schematic of her reasoning.)

This interpretation of Arielle's reasoning makes sense of her seemingly contradictory statement from the second set of quotations: "Yeah, that's the problem, I was thinking they were the same." In the first line of that set of quotations she indicates that she is uncertain about the conclusion that the change in momenta would be the same. We propose that at this time she started to search through her mind for different reasoning strategies that she could employ to corroborate the conclusion that the change in the momenta would be the same. Feature analysis could be a possible reasoning strategy that was tacitly cued. If one reasons with feature analysis, the only way the momenta could be the same is if the change in velocities were also the same. This may be why she claims "I was thinking they were the same," even though in the first set of quotes she says "the change in velocity [for A] has got to be sort of bigger."

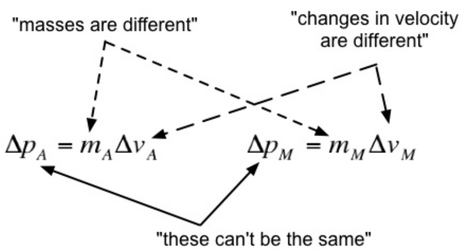

Fig. 3: Interpretation of equation using the intuitive interpretive device feature analysis.

In our videotapes we have frequently seen students in this population invoking feature analysis to interpret an equation and conclude that two quantities are different in places where it would be more appropriate to invoke compensation or balancing and to conclude that the quantities are the same.

Control Structures: Epistemic Games

One of the most interesting characteristics of the student behaviors we observed on tape was their local coherence. Over a period of a few minutes to a half an hour, students were observed to reason using a limited set of associated reasoning tools. An appropriate set of structures for describing these behaviors is the *epistemic game* (or, *e-game*, for short) introduced by Collins and Ferguson.[7] They define the epistemic game as the complex "set of rules and strategies that guide inquiry" and the epistemic form as the (external) "target structure that guides scientific inquiry." The difference between these two concepts is best articulated by Collins and Ferguson:

> *The difference between forms and games is like the difference between the squares that are filled out in tic-tac-toe and the game itself. The game consists of rules, strategies, and different moves that players master over a period of time. The squares form a target structure that is filled out as any particular game is played.*[72]

Collins and Ferguson introduced epistemic games to describe expert scientific inquiry across disciplines. The students in introductory physics courses are far from experts, so using scientists' approaches to inquiry as a norm by which to describe students' inquiry would not be appropriate. For this reason, we generalize the idea of epistemic games to be descriptive rather than normative. We define an *epistemic game* to be[9]

> *a coherent activity that uses particular kinds of knowledge and processes associated with that knowledge to create knowledge or solve a problem.*

The activities are 'epistemic' in the sense that students engage in these activities as a means of constructing new knowledge. We use the word 'game' in a very real sense; a particular game (like checkers or chess) is a coherent activity that has an ontology (players, pieces, and a playing board) and a structure (a beginning and an end, moves, rules) that makes it distinguishable from other activities or games. In the same way, a particular e-game has an ontology and structure that makes it distinguishable from other activities or e-games.

To clarify the ontology and structure of epistemic games, we use the simplest epistemic game identified by Collins and Ferguson: *list making*. Table 4 summarizes the ontological and structural components of epistemic games.) Every list is implicitly an answer to a question. Some examples are: "What do I need from the grocery store?"; "What are the fundamental forces of nature?"; and, "What are the constituents of all matter?"

Note that the idea of a "game" here, a locally coherent set of behavioral rules for achieving a particular goal, is very general. Some of the behavioral science literature (especially in the opposite extremes of popularizations[73] and mathematical economics[74]) has used the term game in this way. When looked at this way, it is obvious that people are often "playing games" in this sense. We are focusing here on epistemic games: games engaged in for the purpose of creating knowledge.

Ontology of Epistemic Games

Epistemic games have two ontological components: the knowledge base and the epistemic form. An e-game is not simply a cognitive structure; it is an activation of a pattern of activities that can be associated with a collection of resources. The collection of resources that an individual draws on while playing a particular e-game

Ontological Components		Structural Components	
Knowledge Base	Cognitive resources associated with the game.	*Entry and ending conditions*	Conditions for when to begin and end playing a particular game.
Epistemic Form	Target structure that guides inquiry.	*Moves*	Activities that occur during the course of an e-game.

Table 4. The ontological and structural components of epistemic games

constitutes the knowledge base. To answer a question like, "What are the fundamental forces of nature?" one needs to have some requisite knowledge to list the forces. The knowledge base for the e-games we identify below includes the resources that we introduced above: intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices.

The *epistemic form* is a target structure that helps guide the inquiry during an epistemic game. For example, the epistemic form in the list making game is the list itself. The list is an external representation that cues particular resources and guides the progression of the inquiry.

Structure of Epistemic Games.

The structural components of epistemic games include the entry and ending conditions of the game and the moves. The entry and ending conditions specify the beginning and the ending of the game. As we mentioned above, one may enter into the list making game as a means to answer a question. When solving physics problems, students' expectations about physics problems determine the entry and ending conditions. These expectations can depend on real-time categorizations of physics problems and/or on preconceived notions about the nature of problem solving in physics. Research by Hinsley and Hayes indicates that students can quickly categorize large classes of physics problems very shortly after reading the statement of the problem.[75] (Often these categorizations can be made after reading the first sentence.) The students' ability to very quickly categorize physics problems may stem from their experience with and expectations about physics problem solving. These expectations and categorizations of physics problems affect which epistemic game the students (perhaps tacitly) choose to play. In addition, students' preconceived epistemological stances about problem solving in physics can affect their expectations. If students believe that problem solving in physics involves rote memorization of physics equations, it can affect what strategy they employ (i.e. which e-game they choose to play) and what they believe an answer in physics is (i.e. how they know they are done playing a particular game).

The second structural component of an epistemic game is the allowed moves: the steps/procedures that occur in the game. In the list-making game the moves may be to add a new item, combine two (or more) items, substitute an item, split an item, and remove an item. As we will see, a critical element of an epistemic game is that playing the game specifies a certain set of allowed moves. What is particularly important about this is not just the moves that are included in the game, but also the moves that are excluded.

Epistemic games students played in introductory, algebra-based physics.

In this section we discuss some of the epistemic games that account for the different problem-solving strategies seen in our data. The games were determined by a semi-phenomenographic approach.[76] We observed a subset of the data and identified what naturally appeared to be coherent and consistent activities as epistemic games. During weekly meetings of the research team (the authors plus other members of the University of Maryland Physics Education Research Group), the transcription and coding of the episodes were scrutinized and the descriptions of the proposed e-games refined. Finally, two different coders independently analyzed a sample episode in terms of epistemic games, with an inter-rater reliability of 80%. After discussion, the two codings were in complete agreement.

We identify six epistemic games that span the different problem-solving approaches seen within our data. (See table 5.) We do not claim that this list spans all possible problem-solving approaches that could be employed during problem solving in physics and we do not claim to identify all possible moves within each game. If we had examined a different population of students or a different domain, it is possible that the list of epistemic games would be different, though we expect some of the games identified here to have broad applicability. We present them as examples of the type of structure we are proposing. In the next section we give a case study showing how analyzing student behavior in terms of these games helps make sense of what they do and do not do in the context of solving a specific problem.

List of epistemic games
Mapping Meaning to Mathematics
Mapping Mathematics to Meaning
Physical Mechanism Game
Pictorial Analysis
Recursive Plug-and-Chug
Transliteration to Mathematics

Table 5. List of epistemic games identified in our data set

Each of these games is described in more detail below. For each epistemic game we give a brief introduction, discuss its ontology and structure, and then we give an example of students playing that game. Note that some of the games have common moves and one game may look like a subset of another. We identify them as distinct games because they have different ending conditions; students playing different games decide they are "done" when different conditions are met.

<u>Mapping Meaning to Mathematics</u>: The most intellectually complex epistemic game that we identify is Mapping Meaning to Mathematics. Students begin from a conceptual understanding of the physical situation de-

scribed in the problem statement, and then progress to a quantitative solution. We identify five basic moves (see Figure 4): (1) develop a story about the physical situation, (2) translate quantities in the physical story to mathematical entities, (3) relate the mathematical entities in accordance with the physical story, (4) manipulate symbols, and (5) evaluate and interpret the solution.

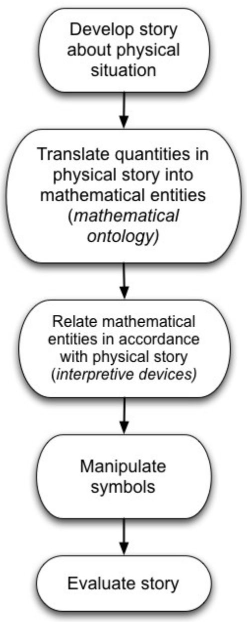

Fig. 4: Schematic diagram of some moves in the epistemic game Mapping Meaning to Mathematics

The knowledge base for this game (as with all the games we identify) comes from the set of physics and mathematics resources; however, in general, different resources can be activated during the different moves of the game. During the development of the conceptual story (move 1), reasoning primitives are most often activated. That is, students often rely on their own conceptual understanding to generate this story – not on fundamental physics principles. Translating the conceptual story into mathematical entities (move 2) is difficult for most of the students in our population. Intuitive mathematics knowledge, symbolic forms, and interpretive devices may be activated during this move. Relating the mathematical entities to the physical story (move 3), again is difficult for students in our population, and relies on intuitive mathematics knowledge, symbolic forms, and interpretive devices. Once the physics equations are written, the symbolic manipulations (move 4) often are carried out without a hitch; most of our students have had ample practice manipulating symbols. The evaluation of the story (move 5) can occur in many different ways: checking the solution with a worked example (or solution in the back of the book), checking their quantitative answer with their conceptual story, or checking their solution against an iconic example.

The epistemic form for *Mapping Meaning to Mathematics* is typically the collection of mathematical expressions that the students generate during moves (2) and (3). These expressions lead the direction of the inquiry. Note, however, that the form is not the entire story in this game. The interpretation (story) that goes with the series of mathematical expressions generated, may or may not be explicitly expressed, depending on the instructions for giving a written output and the students' sense of how much "explanation" they are required to provide.

<u>Mapping Mathematics to Meaning</u>: The ontological components of Mapping Mathematics to Meaning are the same as those in Mapping Meaning to Mathematics. Both games involve the same kind of knowledge base (mathematical resources) and the same epistemic form (physics equations). However, the particular resources and physics equation that are used in each game can vary from problem to problem.

In addition, the structural components of the two games are different. In *Mapping Mathematics to Meaning* students begin with a physics equation and then develop a conceptual story.[77] In the *Mapping Meaning to Mathematics*, students begin with a conceptual story and then translate it into mathematical expressions. The structural differences between these two games make them distinguishable from each other.

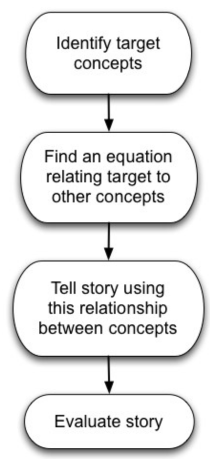

Fig. 5:. Schematic diagram of some moves in the epistemic game Mapping Mathematics to Meaning.

We identify four moves in this game (see figure 5): (1) identify target concept(s), (2) find an equation relating target to other concepts, (3) tell a story using this relationship between concepts, and (4) evaluate story.

<u>Physical Mechanism Game</u>. In the Physical Mechanism Game students attempt to construct a physically coherent and descriptive story based on their intuitive

sense of physical mechanism. The knowledge base for this game consists of reasoning primitives. In this game students do not make explicit reference to physics principles or equations.

The ontology of the *Physical Mechanism Game* is different than in *Mapping Meaning to Mathematics* and *Mapping Mathematics to Meaning*. The epistemic form in the latter two games explicitly involves physics equations; however the epistemic form in the *Physical Mechanism Game* does not. Although the epistemic form is necessarily different, the same set of resources (intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices) may be active in this game as in the previous games.

The structure of the *Physical Mechanism Game* is similar to the first move in *Mapping Meaning to Mathematics* – both involve the development of a conceptual story. However, we can distinguish the two because the *Physical Mechanism Game* represents a separate, coherent unit of student activities; it has a different endstate. In *Mapping Meaning to Mathematics*, after move (1) students go on to move (2), then move (3), etc. After (1) creating and (2) evaluating the conceptual story developed in the *Physical Mechanism Game* (see figure 6) students decide they are done. The activities that follow this game do not cohere with the conceptual story – in direct contrast with the activities that follow move (1) in *Mapping Meaning to Mathematics*.

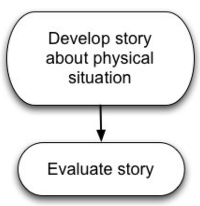

Fig. 6: Schematic diagram of some moves in the epistemic game Physical Mechanism.

<u>Pictorial Analysis Game</u>. In the *Pictorial Analysis Game,* students generate an external spatial representation that specifies the relationship between influences in the problem statement. Students that make a schematic drawing of a physical situation, a free-body diagram, or a circuit diagram are all playing the *Pictorial Analysis Game*.

In this game, as with all the games previously discussed, the knowledge base consists of all the resources listed above (plus some representational translation resources that we do not go into here). The epistemic form in this game is the distinguishing characteristic. The epistemic form is a schematic or diagram that the students generate. For example, if the students draw a circuit diagram during their inquiry, then that diagram serves as an epistemic form which guides their inquiry; in the same way, a schematic drawing or free-body diagram could both serve as target structures that guide inquiry.

The moves in this game are largely determined by the particular external representation that the students choose. For example, if the students choose to draw a free-body diagram, then one move is to determine the forces that act upon the object in question; whereas, if the students choose to draw a circuit diagram, then one move is to identify the elements (e.g. resistors, capacitors, batteries, etc.). There are three moves that are common to all instantiations of the Pictorial Analysis Game (see figure 7): (1) determine the target concept, (2) choose an external representation, (3) tell a conceptual story about the physical situation based on the spatial relation between the objects, and (4) fill in the slots in this representation. An example of students who choose to draw a free-body diagram while playing the Pictorial Analysis Game is given in our example in section 5.[78]

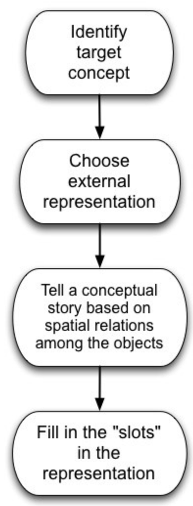

Fig. 7: Schematic diagram of some moves in the epistemic game Pictorial Analysis

<u>Recursive Plug-and-Chug</u>. In the *Recursive Plug-and-Chug Game* students plug quantities into physics equations and churn out numeric answers, without conceptually understanding the physical implications of their calculations.

Students do not generally draw on their intuitive knowledge base while playing this game; they simply identify quantities and plug them into an equation. Therefore, students usually just rely on their syntactic understanding of physics symbols, without attempting to understand these symbols conceptually. That is, their other cognitive resources (intuitive mathematics knowl-

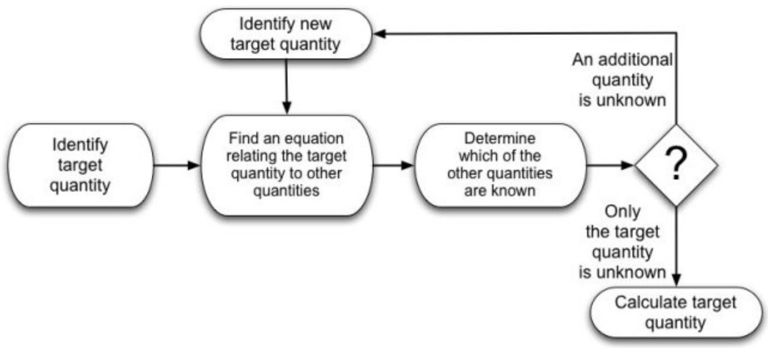

Figure 8. Schematic diagram of some moves in the epistemic game *Recursive Plug-and-Chug*

edge, reasoning primitives, symbolic forms, and interpretive devices) are usually not active during this game.

The epistemic form in *Recursive Plug-and-Chug* is similar to that in *Mapping Meaning to Mathematics* and *Mapping Mathematics to Meaning*. Each game has physics equations as part of the epistemic form, but the resources that are active (*i.e.* knowledge base) in *Recursive Plug-and-Chug* are different than in these other games. Therefore, the rules and strategies that are employed during this game differ from those in *Mapping Meaning to Mathematics* and *Mapping Mathematics to Meaning* – even though the epistemic form (target structure that guides inquiry) is the same in all these games. A distinguishing feature of *Recursive Plug-and-Chug* is the resources that are not activated during this game.

Because the epistemic forms are similar, the structure of *Recursive Plug-and-Chug* is superficially similar to *Mapping Mathematics to Meaning*. First, the students identify a target quantity. This is similar to the first move in *Mapping Mathematics to Meaning*, but it differs in that here the students only identify the quantity and its corresponding symbol – they do not attempt to understand conceptually what the quantity represents physically. Second, the students identify an equation that relates the target quantity to other quantities, but they do not attempt to create a story that justifies the use of that equation. Third, the students identify which quantities are known and which quantities are unknown. If the target quantity is the only unknown, then they can proceed to calculate the answer. However, if there are additional unknowns, then they must choose a sub-goal and start this process over. Herein lies the 'recursive' nature inherent in this game. Figure 8 shows a schematic depiction of the moves in this game.

Transliteration to Mathematics. Research on problem solving indicates that students often use worked examples to develop solutions to novel problems.[79,80] Transliteration to Mathematics is an epistemic game in which students use worked examples to generate a solution, yet they do so without developing a conceptual understanding of the worked example. 'Transliterate' means "to represent (letters or words) in the corresponding characters of another alphabet."[81] In the Transliteration to Mathematics game students simply map the quantities from a target problem directly into the solution pattern of an example problem.

Because students use the symbolism in this game without conceptual meaning, usually only resources associated with the syntactic structure of equations are active during this game. The solution pattern of the target example serves as the epistemic form for the *Transliteration to Mathematics* game.

The moves in this game are simple: (1) identify target quantity, (2) find a solution pattern that relates to the current problem situation, (3) map quantities in current problem situation into that solution pattern, and (4) evaluate the mapping (see figure 9). Many students find moves (2) and (3) very tricky. Many times students may find a solution pattern that they think relates to the current problem, when in fact it does not.

Note that these games are generic structures; they do not specify explicitly what to do, they specify the kind of activity to do. In order for the moves to become explicit, they have to be mapped onto particular realizations that depend on the specific content: both the particular problem involved and the field of knowledge to which the problem is perceived as belonging. In a less neural-based model, such structures might be seen as patterns ("schemas") with "slots" into which particular bits of knowledge are inserted. In our neural-based model, a bit of knowledge is not conceived as a "token" that can be moved around; rather, it is conceived of as a linkage pattern in a neural net that either can be activated or not.[82] As a result, the choice to activate a particular combination of knowledge elements in a particular way is a control element and is presumably carried out by the activation of a network of control neurons that activates the particular knowledge needed. This is why we identify epistemic games as a control structure rather than as an associational structure of knowledge elements.

5. Analysis of a Specific Case

We now present a case study that demonstrates how an analysis in terms of resources and e-games can help make sense of student problem-solving behavior; in particular, why students often don't use what seems to the instructor to be the appropriate resources in a given context.

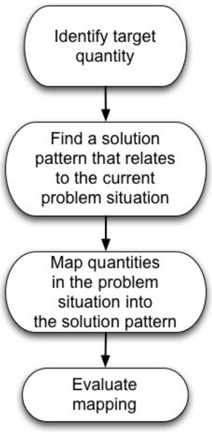

Fig. 9: Schematic diagram of some moves in the epistemic game Transliteration to Mathematics

The episode for this case study involves three female students working on an electrostatics problem: The Three Charge Problem (Appendix, problem #4).[83] This episode occurs in the first week of the second semester of the two-semester introductory, algebra-based physics course. All the students in the group had been in the reformed course the first semester and were familiar with its innovative features. In particular, they were familiar with the interaction style between students and teaching assistants in the course center, and the type of homework problems that were assigned in this course. Most importantly, they were cognizant of the fact that the instructor expected the students to spend about an hour on each homework problem – during which time they were expected to generate solutions to the questions that 'made sense to them.'

An "instructor's" solution to problem #5 involves straightforward balancing of forces and the use of Coulomb's Law. The parenthetic comment in the problem states there is "no net electrostatic force" acting on charge q_3. Symbolically, this becomes $\vec{F}_{q_2 \to q_3} + \vec{F}_{q_1 \to q_3} = 0$. Manipulating this equation, and defining the positive $\hat{i}$ direction to be to the right, yields:

$$\begin{aligned} \vec{F}_{q_2 \to q_3} &= -\vec{F}_{q_1 \to q_3} \\ \frac{kq_2 q_3}{d^2}\hat{i} &= -\frac{kq_1 q_3}{(2d)^2}\hat{i} \end{aligned} \qquad (1)$$

Canceling similar terms on both sides of the equation and setting $q_2 = Q$ yields the result: $q_1 = -4Q$.

There are several inferences and steps involved in generating this solution. However, in spite of the multiple steps involved, most experienced physics teachers solve this problem in less than one minute. Some can "see" the answer in a conversational beat and give the correct answer immediately. The most interesting aspect about the students' approach is that it takes so long. The students work for nearly 60 minutes before arriving at a solution – two orders of magnitude longer than the typical teacher! Why does it take so long? According to the theoretical framework developed here, the typical teacher probably has a broader mathematical knowledge base (*i.e.* a larger collection of compiled mathematical resources) and richer collection of problem solving strategies (*i.e.* an assortment of epistemic games for solving problems in physics). For the typical teacher, the problem statement immediately cues the appropriate epistemic game and tightly compiled resources; whereas, the students' mathematical resources do not exist in compiled form. The difference in the teacher and the students' knowledge structure could account for the difference in the speed of the problem solution and demonstrates the power and effectiveness of cognitive compilation.[84]

The students do not follow a straightforward approach to solving this problem. However, these students' various problem-solving approaches are easily understood in terms of epistemic games. We identify five different epistemic games that are played during this problem solving session: *Physical Mechanism*, *Pictorial Analysis*, *Mapping Mathematics to Meaning*, *Transliteration to Mathematics*, and *Mapping Meaning to Mathematics*. We divide the discussion into segments corresponding to different e-games and refer to these segments as "clips."

Playing the Physical Mechanism Game

The students' initial attempt to solve this problem follows a less formal path than the instructor's solution outlined above. Throughout this clip the students draw on intuitive reasoning primitives to explain and support their conclusions. The students do not activate any formal mathematics or physics principles to support their claims. The reasoning consists almost entirely of facets. This first clip occurs about 7 minutes into the problem-solving process.

Darlene: *I'm thinking that the charge q_1 must have it's...negative Q.*

Alisa: *We thought it would be twice as much, because it can't repel q_2, because they're fixed. But, it's repelling in such a way that it's keeping q_3 there.*

Bonnie: *Yeah. It has to—*

Darlene: *Wait say that.*

Alisa: *Like— q_2 is— q_2 is pushing this way, or attracting—whichever. There's a certain force between two Q, or q_2 that's attracting.*

Darlene: *q_3.*

Alisa: *But at the same time you have q_1 repelling q_3.*

Darlene initiates the conversation by asserting that the charge on q_1 must be 'negative Q'; the negative sign in this case standing for her realization that q_1 and q_2 will have opposite effects on q_3. Alisa elaborates on this point by articulating that q_2 exerts an influence on q_3, which she identifies as a force, that is either repelling or attracting, and that q_1 exerts the opposite influence on q_3. The semantic content contained in Alisa's explanation can be summarized in the following facet: 'the attractive effect of q_2 on q_3 cancels the repulsive effect of q_1 on q_3.' The abstract reasoning primitive underlying this facet is *canceling*. In this case, *canceling* is an appropriately mapped primitive, because in fact the two forces acting on q_3 do cancel, which results in there being no net electrostatic force on q_3.

From Alisa's initial cursory comment ("we thought [the charge on q_1] would be twice as much [as the charge on q_2]") it appears that she has the reasoning primitives *more is more* and *balancing* activated. That is, since the two influences acting on q_3 balance, then q_1 must have more charge because there is more distance between q_1 and q_3 then there is between q_2 and q_3.

It cannot be confirmed whether Alisa has *more is more* and *balancing* activated, because the direction of the conversation changes. Darlene contends with the other students, because it appears she has activated a different reasoning primitive: *blocking*.

Darlene: *How is it repelling when it's got this charge in the middle?*

Alisa: *Because it's still acting. Like if it's bigger, than q_2 it can still, because they're fixed. This isn't going to move to its equilibrium point. So, it could be being pushed this way.*

Darlene: *Oh, I see what you're saying.*

Alisa: *Or, pulled. You know, it could be being pulled more, but it's not moving.*

Darlene: *Um-huh.*

The orientation of the charges cues the reasoning primitive of *blocking*, because q_2 is between q_1 and q_3. From the superposition principle we know the effect of q_1 on q_3 does not get blocked by the presence of q_2, so the activation of *blocking* is an unnecessary distraction. In contrast to the reasoning primitive of *canceling* that was activated earlier in this clip, *blocking* does not get mapped into a productive facet for solving this problem. (This is not to say that *blocking* is 'wrong'; rather, in this particular instance the activation of *blocking* does not lead to a productive facet.)

Bonnie continues Alisa's line of reasoning by explaining why the value of q_1 has to be twice as big as that of q_2.

Alisa: *So, we—we were thinking it was like negative two Q or something like that.*

Bonnie: *Yeah. Cause it has to be like big enough to push away.*

Darlene: *Push away q_3.*

Bonnie: *Yeah, which we—which I figured out negative two.*

Darlene: *Cause it's twice the distance away than q_2 is?*

Bonnie: *Yeah.*

Darlene: *I agree with that.*

It appears that Alisa draws on *overcoming* when she explains that 'Like if it's bigger, than q_2 it can still [have an effect]' and Bonnie restates this as '[q_1] has to be like big enough to push away [q_3].' That is, q_1 has to have enough charge to overcome the influence of q_2. The tacit conclusion from this assertion is that the charge of q_1 must have a larger magnitude than that of q_2. This is particularly interesting since Alisa later shows (see below) that she understands Coulomb's law and superposition. But in the context of *Physical Mechanism* she generates an (incorrect) argument in support of her sense that the force from both source charges must be included using reasoning primitives and facets.

Bonnie and Darlene quantify this conclusion by using the reasoning primitives of *more is more* and *dependence* (which has the symbol template $\square = [\ldots x \ldots]$) to assert that the charge on q_1 has to be twice the magnitude of q_2. *More is more* and *dependence* get mapped into the facet *twice the distance is twice the charge*. Bonnie's argument stays within the rules of the local e-game. Because *physical mechanism* does not include moves that access formal knowledge, they do not invoke the formal knowledge that says *blocking* is irrelevant. We will see later that they (and Alisa in particular) indeed do have the relevant formal knowledge.

The students' problem solving activities during this entire clip have the ontology of *Physical Mechanism*. While playing this game the students draw on their intuitive knowledge base rather than their formal knowledge to support their claims. During this clip the students use various reasoning primitives and do not mention any formal mathematics or physics principles. The epistemic form in the *Physical Mechanism Game* involves a coherent, physical description that is either verbal or imagistic. These students are actively seeking physical causes for the effects that are described in the problem.

Playing this game helps the students become oriented to this problem, but the solution to this problem necessarily involves physics equations (in particular Coulomb's Law). Since *Physical Mechanism* does not include mathematical expressions or equations it cannot ultimately lead them to the correct answer. In the next clip, a comment from the TA helps them reframe the problem, activate other resources they have, and play another epistemic game.

Playing the Pictorial Analysis Game

In the last clip we saw the students making sense of the problem by using their intuitive reasoning primitives in the context of Physical Mechanism game. At the end of the clip, the students appear to have difficulty focusing their collective attention. To assist them, the TA (Tuminaro) offers a suggestion.

Darlene: I think they all have the same charge.

Bonnie: You think they all have the same charge? Then they don't repel each other.

Darlene: *Huh?*

Bonnie: *Then they would all repel each other.*

Darlene: *That's what I think is happening.*

Bonnie: *Yeah, but q_3 is fixed. If it was being repelled—*

Alisa: *No, it's not. q_3 is free to move.*

Bonnie: *I mean, q_3 is not fixed. That's what I meant.*

Darlene: *Right.*

Bonnie: *So, like...*

Darlene: *So, the force of q_2 is pushing away with is only equal to d.*

Bonnie: *Yeah, but then...*

Darlene: *These two aren't moving.*

Bonnie: *Wouldn't this push it somewhat?*

Alisa: *Just because they're not moving doesn't mean they're not exerting forces.*

Darlene: *I know.*

Alisa: *What do you think?*

TA: *Can I make a suggestion?*

Darlene: *Uh-huh.*

TA: *You guys are talking about like a lot of forces and stuff. And, one thing I've suggested in previous semesters, if you write it down and say, what forces do you think are acting here, you can all talk about it.*

Darlene: *Where did the marker go?*

TA: *That's a suggestion—a general suggestion—that I might make.*

In the first few lines above, it seems as though the students take a step back. Earlier, they appeared to have established the major aspect of the problem: two influences act on q_3, which exactly cancel each other. In this clip, the students restate the set up of the problem ("these two are moving") and recite remembered facts ("just because they're not moving doesn't mean they're not exerting forces"). While these things are important to keep straight, this discussion does not appear to push the problem-solving process forward.

The suggestion to write on the whiteboards has two effects on the students. First, it nudges them into playing a different epistemic game, *pictorial analysis*.[85] Second, the introduction of this new epistemic game and a new e-form reframes the students' interactions and helps them focus their collective attention and clarifies their communication.

Alisa attempts to make an external representation of this problem on the white board while Bonnie and Darlene offer their assistance:

Darlene: *You're trying to figure out what q_1 is, right?*

Bonnie: *Oh, yeah.*

Alisa: *Because this is in equilibrium, there's some force...*

Darlene: *Pulling it that way and some force pulling ex—equally back on it.*

Bonnie: *Yeah.*

Alisa: *And, they're equal?*

Bonnie: *Yes.*

Darlene: *Same with up and down. Not that that matters, really.*

Bonnie: *We'll just stick with...*

Darlene: *Horizontal.*

Bonnie: *Yeah, one dimension.*

In this clip the students are deciding which features mentioned in the problem should be included in their diagram — a move within *Pictorial Analysis*. The structure of this game is similar to *Physical Mechanism*; however, the ontological components of *Physical Mechanism* and *Pictorial Analysis* are different. The epistemic form in *Pictorial Analysis* involves a coherent, physical description *and* an external representation; the epistemic form for *Physical Mechanism* only involves a coherent, physical description.

The external representation generated in the *Pictorial Analysis* epistemic game activates additional resources in the students, which help them better understand this problem. In particular, the students draw on the interpretive device of *physical change* to conclude that q_1 and q_2 have to have opposite charges.

Alisa: *So, maybe this is pushing...*

Darlene: *That's [q_2] repelling and q_1's attracting?*

Bonnie: *Yeah, it's just that whatever q_2 is, q_1 has to be the opposite. Right?*

Alisa: *Not necessarily.*

Darlene: *Yeah.*

Bonnie: *OK, like what if they were both positive?*

Alisa: *Well, I guess you're right, they do have to be different, because if they were both positive...*

Bonnie: *Then, they'd both push the same way.*

Alisa: *And, this were positive it would go zooming that way.*

Darlene: *They would both push.*

Alisa: *And, if this were negative it would go there.*

Bonnie: *It would go zooming that way.*

Alisa: *And, if they were negative...*

Darlene: *It would still—they'd all go that way.*

Alisa: *It would be the same thing.*

Bonnie makes a claim that the charge on q_1 has to be the opposite of q_2, but the others don't initially agree, despite the fact that they had mentioned the result in the context of a different game in the previous clip. Bonnie's suggestion to verify, or falsify, her claim involves the interpretive strategy of *physical change*. That is, she considers the affect of an actual physical alteration to the system ("OK, like what if they were both positive?"). From this move the students almost immediately conclude that the charges on q_1 and q_2 must be different, or else q_3 would go 'zooming' away.

Switching to *Pictorial Analysis* turns out to be a very effective strategy for this group of students. By decomposing the forces in space and creating an external representation, they are able to physically justify why q_1 and q_2 have to have opposite charge. This clip also illustrates that the students' problem does not stem from *lack* of knowledge or skills; rather, the epistemic game the students play in their initial approach (*Physical Mechanism*) does not help adequately articulate the physical relationship between the charges. The external representation they collectively generate in *Pictorial Analysis* cues resources they already possess (*physical change*), which helps them make progress on this problem (i.e. conclude with confidence that q_1 and q_2 have opposite charges).

Although the students' external representation and conclusion marks progress, they have yet to solve the problem. In fact, they have not even identified the necessary physics principle: *Coulomb's Law*. That's what happens in the next clip.

Playing the Mapping Mathematics to Meaning Game

So far the students have drawn a diagram representing which forces act and in what direction, and they have concluded that q_1 and q_2 have opposite charges; however, they have not yet solved the problem. In this clip we see Alisa spontaneously reframe the problem solving process by drawing on a new set of resources: *formal mathematics knowledge*.

Alisa: *Are we going to go with that?*

Bonnie: *I think it makes sense.*

Darlene: *That makes...*

Alisa: *Well, I don't know, because when you're covering a distance you're using it in the denominator as the square.*

Bonnie: *Oh! Is that how it works?*

Alisa: *And (?) makes a difference.*

Bonnie: *Yeah, you're right.*

TA: *So, how do you know that?*

All: *From the Coulomb's Law.*

Bonnie: *So, it should actually be negative four q? Or what? Since it has…*

Alisa: *Cause we were getting into problems in the beginning of the problem with* [the force-distance two-charge problem] (Appendix, problem #5)*, because I thought that like if you move this a little bit to the right the decrease for this would make up for the increase for this. But, then we decided it didn't. So, that's how I know that I don't think it would just increase it by a factor of two.*

Alisa is not only attempting to introduce an equation, she is negotiating a shift in how this problem is being viewed — asking the group to play a new epistemic game. All the previous reasoning relied on intuitive reasoning primitives, without any explicit reference to physics principles or equations. Alisa's introduction of Coulomb's Law is the first mention of a physics principle during this entire problem-solving process. In addition, it's the first time anyone explicitly makes reference to an equation ("when you cover a distance you use it in the denominator as the square"). Alisa's use of formal physics principles and explicit reference to equations is (tacitly) asking the other students to play *Mapping Mathematics to Meaning*.

Alisa's discussion follows all the moves within *Mapping Mathematics to Meaning* (see figure 10). First, the distance and force are identified as the relevant concepts in this problem. Second, she identifies Coulomb's Law $\left(F = \frac{kq_1q_2}{r^2}\right)$ as an equation that relates the target concept to other concepts. Third, she develops a story using this relationship between concepts: "When you're covering a distance you're using it in the denominator as the square." Fourth, she evaluates the validity of her story by referencing a previous problem. She acknowledges that her intuitive reasoning had failed her on the previous problem, which justifies for her the need for Coulomb's Law on this problem: "I thought that like if you move this a little bit to the right the decrease for this would make up for the increase for this. But, then we decided it didn't."

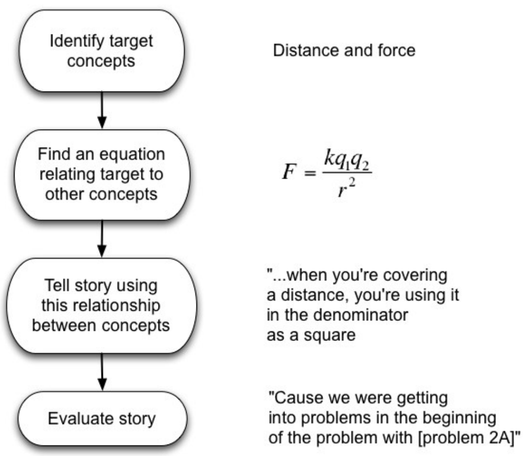

Fig. 10: Schematic map of Alisa's moves within Mapping Mathematics to Meaning

Alisa's use of Coulomb's Law is significant progress on this problem, but all the other students don't know how to apply this new piece of information. In fact, the introduction of Coulomb's Law cues Darlene to play a new (and counter-productive) epistemic game.

Playing the Transliteration to Mathematics Game

Although it appears the students are making progress on this problem, they take a detour and attempt to use another problem as a prototype for solving this problem. Alisa has suggested that Coulomb's Law is an important concept. It appears that Darlene does not initially know how to apply this new information. She attempts to find a different problem that uses Coulomb's Law in its solution, and then map the solution pattern from the other problem to the Three-Charge Problem. The problem that Darlene identifies as using Coulomb's Law in the solution is the Force-Distance Two-Charge Problem (Appendix, problem #5).

Darlene: *Where is that other problem? Three times as far apart as they were now what is the magnitude of the force?*

Bonnie: *I think it should be four times.*

Darlene: *If it's three times as far apart it's...you divide...uh! I think it's q over two.*

Bonnie: *Q over two? So, if you think of it as half the force of* q_2.

Darlene: *Look at this one.*

Bonnie: *Is this one you're talking about?*

Darlene: *Uh-huh. If you increase the distance that they are from each other it's decreasing by the same amount. I thought it was four (?), but they said it was (?). I don't know why. Just three times...does it matter? I'm looking at this one. Number three, isn't that like the same thing?*

Alisa: *Three was an estimation problem.*

Darlene: *No, no with the q and four q and all that, you know how there was this question that asked when you move the charges three times further apart than they originally were, what the resulting force is.*

Alisa: *OK.*

Darlene: *And, you said it was—we said it was four—the charge would be like q, or nine, but it would get three times as far apart. Why it's not three I don't understand, but that's all right. So—*

Alisa: *Well, 'cause in the equation you square this—the distance between them. Like if you're multiplying by three...*

Darlene: *Oh! So, I would think this one would be q over four—negative q over four. Cause it's twice as far away, opposite charge. Does that make sense?*

Alisa: *But, then it's a smaller charge than this.*

Bonnie: *Yeah.*

Alisa: *So, I don't understand how it would be pushing three or pulling three whatever it's doing.*

In the Force-Distance Two-Charge Problem, the students had found that if the force between two charges for a given distance is *F*, tripling the distance results in a force between the two charges that is decreased by a factor of nine (see Appendix, problem #5), in compliance with Coulomb's Law. Darlene is attempting to match the quantities in the Three Charge Problem with quantities from the Force-Distance Charge Problem, so the solution pattern can be transferred; *i.e.* she is playing the *Transliteration to Mathematics* epistemic game.

One obvious piece of evidence that Darlene is playing *Transliteration to Mathematics* comes when she says, "Why it's not three I don't understand, but that's all right." Darlene is explicitly indicating that she doesn't understand the previous problem, but conceptual understanding is not a move in the *Transliteration to Mathematics* epistemic game. All that is important is that the problems have enough similar features that the solution from one problem can be transferred to the other.

Darlene's metacognitive statement ("Why it's not three I don't understand, but that's all right.") stands in stark contrast to Alisa's meta-cognitive statement ("I thought that like if you move this a little bit to the right the decrease for this would make up for the increase for this."). Darlene simply admits she doesn't understand and slavishly transfers the solution pattern from the previous problem anyway. Alisa's metacognitive statement leads to her justification for using Coulomb's Law.

Darlene's *Transliteration to Mathematics* approach doesn't help her with the Three-Charge Problem. She says, "If you increase the distance that they are from each other it's decreasing by the same amount." The problem with Darlene's approach is that she is unaware of the two meanings that she attributes to the pronoun 'it.' In the previous problem the pronoun stands for 'force,' so that the statement would read, "If you increase the distance that they are from each other, then *the force* is decreasing by the same amount." However, Darlene tacitly maps this into the statement, "If you increase the distance that they are from each other, then *the charge* is decreasing by the same amount." The *Transliteration to Mathematics* game is not helpful in this case because force and charge are not related to distance in the same way in Coulomb's Law. The charge has to be found from the balance of two forces. This is not to say that the *Transliteration to Mathematics* game is wrong; it doesn't work in this situation because of Darlene's inappropriate mapping of force and charge. The components (ontology and structure) of *Transliteration to Mathematics* can also be found as a part of the richer *Mapping Mathematics to Meaning* game – just as was true for *Recursive Plug-and-Chug*, but the goals (perceived endstates) of the games differ.

Playing the Mapping Meaning to Mathematics Game

In this clip the students finally come to the solution of the problem. Alisa summarizes her final solution as the other students listen. Alisa's problem solving activities follow the Mapping Meaning to Mathematics epistemic game (see figure 11). First, she develops a conceptual story describing the physical situation. This conceptual story relies heavily on the reasoning primitives of balancing.

TA: *What did you do there?*

Alisa: *What did I do there?*

TA: *Yeah, can I ask?*

Alisa: *All right, so **because this isn't moving, the two forces that are acting on it are equal**: the push and the pull.*

Alisa correctly maps 'force' as the two influences that balance in this physical situation. *Second*, Alisa uses the *identity* symbolic form, which has the symbol template ☐ = …, to translate her conceptual story into mathematical expressions:

Alisa: *So, the F—I don't know if this is the right F symbol—but, the F q_2 on q_3 **is equal to** this (see Equation 2). And, then the F q_1 on q_3 **is equal to** this (see Equation 3), because the distance is twice as much, so it would be four d squared instead of d squared.*

$$F_{q2\to q3} = \frac{kQq_3}{d^2} \quad (2)$$

$$F_{q1\to q3} = \frac{kxQq_3}{4d^2} \quad (3)$$

Alisa explains why she wrote the charge on q_1 as '*xQ*,' by drawing on the reasoning primitive of *scaling*, which has the syntax *x*☐.

Alisa: *And, then I used xQ like or you can even do—yeah—xQ for the charge on q_1, because we know in some way **it's going to be related to Q** like the big Q we just got to find **the factor that relates** to that.*

In the third step in the *Mapping Meaning to Mathematics*, Alisa relates the mathematical entities that she derived in step 2 with her conceptual story that she developed in step 1:

Alisa: *Then, I set them equal to each other…*

Fourth, she manipulates the mathematical expression to arrive at the desired solution:

Alisa: *... and I crossed out like the q_2 and the k and the d squared and that gave me Q equals xQ over four. And, then Q equals four Q, so x would have to be equal to four. That's how you know it's four Q.*

Fifth, the other students evaluate Alisa's problem solving approach and conclusion.

Bonnie: *Well, shouldn't it be—well equal and opposite, but...*

Alisa: *Yeah, you could stick the negative.*

Bonnie: *Yeah.*

Darlene: *I didn't use Coulomb's equation, I just—but it was similar to that.*

Bonnie: *That's a good way of proving it.*

Darlene: *Uh-huh.*

Bonnie: *Good explanation.*

Alisa: *Can I have my A now?*

Darlene and Bonnie accept Alisa's approach is "a good way of proving it." In fact, Alisa must realize that this is a good way to prove this, since she self-evaluates her solution and asks for an "*A* now."

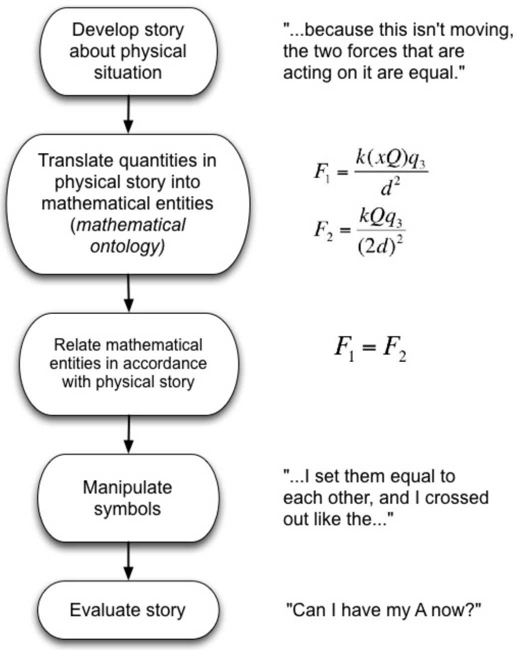

Fig. 11: Schematic map of Alisa's moves within Mapping Meaning to Mathematics.

6. Implications for Instruction, and Conclusions

Far too often in physics instruction, physics teachers focus on the content and results rather than on what it is they really wants their students to learn — how to think about physics. When we choose our content as learning Newton's or Kirchoff's Laws, when we give students a syllabus consisting of particular chapters in an encyclopedic text, and when we permit them to take a card consisting of all the "necessary equations" into an exam, we are sending an unintended message that what matters in physics are the equations and results. Physics teachers generally know that this is not the case and may stress understanding and reasoning in their lectures. But students may not understand what this means and ignore the derivations and reasoning the teacher presents, boxing the final results in their notes for memorization and ignoring the process that generated them. Research in physics education documents that the students bring understandings of the physical world in to a physics class that may contradict and confound their attempt to make sense of what is being taught. This can lead them to emphasize memorization — which in some environments works even when you have little understanding of what you are memorizing.

Physics teachers often have the sense that "problem solving is where you learn to actually do physics" — where students should learn the process and sense making that exemplifies good physical thinking. We assign many problems and model the processes we want them to follow in lectures. But without a good understanding of what kinds of thinking the students need to activate for solving problems, it is difficult to understand why some students seem to get so little out of solving large numbers of problems. The teacher's knowledge exists in compiled form; whereas, the students' knowledge does not. Instructors may not be aware of all the knowledge and reasoning that goes into solving a problem, if the solution comes so easily and quickly. Decomposing the students' problem-solving session in terms of epistemic games and resources allows us to 'see' and examine the knowledge and reasoning that is involved in this problem. With increased understanding of the knowledge and reasoning involved in a seemingly simple problem, instructors and educators can begin to develop teaching environments and interventions that more effectively and efficiently cue the appropriate resources and epistemic games. This in turn could help students become better and more efficient problem solvers.

This study focused on students at the university level and most of them were upper division students who had already taken many science courses at the university. As a result, we make no claims concerning the origin of the games the students were playing. However, we often saw the students playing games that were clearly learned somewhere. Students playing Recursive Plug-and-Chug, for example, had a very strong sense of what they thought they were supposed to be doing — both the goals of their local activities and what they ought to be doing to get there. We can well imagine the students being taught to "identify the variable to be found," to "find an equation containing that variable," and so forth. This is strongly suggestive that instructors need to be aware that when they are teaching their students processes that can produce effective results in situations with a particular limited class of problems to be solved, they may also unintentionally be teaching their students to play particular epistemic games without helping them to develop a good sense of when those games might (or might not) be appropriate. Such instruction might help students get through the vicissitudes of a particular course but might have unintended negative consequences at later stages in the students' education.

This study gives just one example of how cognitive modeling helps increase our understanding of just what it is our students need to learn. The specific resources and games we describe are not meant to be complete, but rather to introduce a new kind of structure for analyzing students' thoughts on problem solving.

Our focus has been on structures in the cognitive model of the individual student, but it is clear that two additional factors play essential roles and require further research. First, the student's decisions (tacit or conscious) about which games to play have a critical role. Second, the interaction of the students in their group games is extremely important and the structures proposed here could be of considerable help in understanding a group's negotiation of how to approach and solve a problem.

Acknowledgement

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Appendix

1. Conversion Problem

Discuss the question: "Is 500 feet big or small?" Before you do so, carry out the following estimates.

(a) You are on the top floor of a 500 ft tall building. A fire breaks out in the building and the elevator stops working. You have to walk down to the ground floor. Estimate how long this would take you. (Your stairwell is on the other side of the building from the fire.)

(b) You are hiking the Appalachian Trail on a beautiful Fall morning as part of a 10 mile hike with a group of friends. You are walking along a well-tended, level part of the trail. Estimate how long it would take you to walk 500 feet.

(c) You are driving on the New Jersey Turnpike at 65 mi/hr. You pass a sign that says "Lane ends 500 feet." How much time do you have in order to change lanes?

2. Air Drag Problem

For the first part of the problem, let's figure out what the drag force has to look like as a function of the possible variables using dimensional analysis. Consider a sphere of radius R and mass m moving through the air at a speed v. Assume the air has a density ρ (measured in kg/m^3)

- The force the air exerts on the sphere is independent of the sphere's mass. Discuss why this is plausible. (*Hint*: consider the case of the sphere held fixed and the air blowing past it at a speed v.)
- From the quantities R, ρ, and v use dimensional analysis to show that there is only one possible combination of these variables that produces a quantity with the dimension of force.

3. Colliding Gliders (Algebraic) Problem

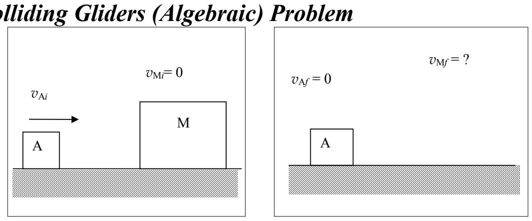

Figure 12. Figure for the colliding gliders problem.

The mass of glider A is one-half that of glider M (i.e. $m_M = 2m_A$). Apply Newton's second law ($F_{net} = m\Delta v/\Delta t$) to each of the colliding gliders to compare the *change in momentum* ($\Delta p = m\Delta v$) of gliders A and M during the collision. Discuss both magnitude and direction. Explain.

4. Three-Charge Problem

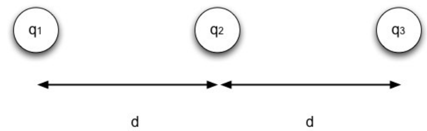

Figure 13. Figure for the three-charge problem.

In the figure above three charged particles lie on a straight line and are separated by distances d. Charges q_1 and q_2 are held fixed. Charge q_3 is free to move but happens to be in equilibrium (no net electrostatic force acts on it). If charge q_2 has the value Q, what value must the charge q_1 have?

5. Two-Charge Problem

Two small objects each with a net charge of Q (where Q is a positive number) exert a force of magnitude F on each other. We replace one of the objects with another whose net charge is $4Q$. If we move the Q and $4Q$ charges to be 3 times as far apart as they were. Now what is the magnitude of the force on the $4Q$?

(a) F/9 (b) F/3 (c) 4F/9 (d) 4F/3 (e) other

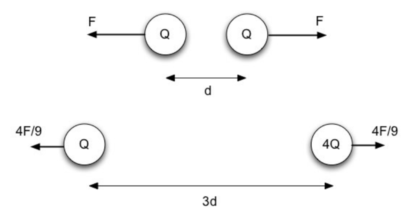

Figure 14. Figure for the force-distance two-charge problem (with the answer shown).

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