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Magnetic dipole transitions in atomic and particle physics: ions and psions†

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Abstract

Magnetic dipole (M1) transitions in atomic systems can be classified into two types. If in a non-relativistic description of the atomic states the spatial parts of the initial and final wavefunctions are the same, as in the hydrogenic transition $2^2P_{3/2} \rightarrow 2^2P_{1/2}+\gamma$, one has an 'ordinary' or 'unhindered' transition. If the spatial parts are orthogonal, as in $2^2S_{1/2} \rightarrow 1^2S_{1/2}+\gamma$ in H, one has a 'relativistic' or 'hindered' transition. The study of such relativistic M1 transitions, initially of concern only for astrophysics, then for solar physics, has come down to earth in the present decade with the advent of beam–foil spectroscopy. After a review of the theory of these decays in H and H-like ions, the history of the development of interest in the analogous transition $2^3S_1 \rightarrow 1^1S_0+\gamma$ in He and He-like ions is sketched and the corresponding theory is developed. In contrast to the hydrogenic case, experimental values for the lifetime of the 2^3S_1 state are now available for many species in the helium isoelectronic sequence ranging from neutral helium with $\tau \simeq 10^4$s to thirty-four-times ionised krypton with $\tau \sim 10^{-10}$ s. A systematic comparison of the theoretical and experimental values, which range over *fourteen* orders of magnitude, is carried out. This provides a new and stringent test for quantum electrodynamics, now beginning its second half-century of existence.

The spectacular discovery, in 1974, of the J/ψ and ψ' particles in high-energy proton–nucleus and electron–positron collisions is reviewed and the theoretical background is given for the far-reaching hypothesis that these particles can be regarded as $^3S_1$ bound states of a 'charmed quark' and its antiparticle. The qualitative success of this assumption, resulting from the subsequent discovery of predicted other narrow resonances in the 3–4 GeV region coupled to the J/ψ and ψ' by one-photon emission, is described and the ideas of the so-called 'naive charmonium model' are outlined. The theory of the radiative transitions of bound states of two spin-$\frac{1}{2}$ particles is then developed in some detail, with special emphasis on magnetic dipole transitions; specific dynamical assumptions are avoided as much as possible. The theory is applied to the charmonium model and the great sensitivity of the relativistic M1 decays, such as $2^3S_1 \rightarrow 1^1S_0+\gamma$ or $2^1S_0 \rightarrow 1^3S_0+\gamma$, to the form of the relativistic

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interaction between the quark and the antiquark is illustrated. A brief survey is given of recent attempts to overcome the deficiencies of the model.

The review concludes with a discussion of other problems of current interest in which relativistic M1 transitions may play an important role.

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Contents

1. Introduction

On 2 February 1927, a paper entitled 'The Quantum Theory of the Emission and Absorption of Radiation' by P A M Dirac (1927) arrived at the editorial office of the *Proceedings of the Royal Society*. Thus, 1977 may be regarded as the golden anniversary of the birth of quantum electrodynamics (QED), the quantum theory of the electromagnetic field in interaction with charged particles. Enormously successful, especially when supplemented by the relativistic theory of the electron and the ideas of renormalisation theory, QED became the prototype for all later attempts to also build a quantum field theory of the weak and strong interactions of elementary particles.

From its beginnings, QED has played a dual role in fundamental physics. On the one hand, it has served as a framework for the theoretical interpretation of a huge variety of experimental facts concerning atoms and molecules, especially those directly related to the emission and absorption of photons. On the other hand, study of the electromagnetic properties (charge, magnetic moment, radiative decays, etc) of nuclei, nucleons, and of the host of unstable 'elementary' particles discovered in the past forty years has provided constraints which any theory of the basic interactions and of the structure of hadrons (strongly interacting particles) has had to meet.

The purpose of this article is to review some recent developments which exemplify applications of quantum electrodynamics to two seemingly quite distinct areas of physics: (i) certain 'forbidden' radiative transitions in the helium isoelectronic series, and (ii) certain radiative transitions involving the very narrow resonant states discovered in November 1974 in high-energy (multi-GeV) experiments at Brookhaven National Laboratory and the Stanford Linear Accelerator Center. In case (i) we shall find QED playing its traditional historical role of correlating phenomena in atomic physics, however, in a domain where it has not been tested previously. In case (ii) we shall see it playing its other major role, of providing a constraint on theories of hadronic structure in particle physics. The concept which unites these topics and motivates a joint review is that of a 'relativistic magnetic dipole transition'. Let us consider in a preliminary way what is meant by such a transition and why such transitions are currently of interest.

Recall first that in atomic physics most radiative transitions are electric dipole (E1) transitions, satisfying the selection rule: '$\Delta J = \pm 1, 0$; there is a change in parity', where ΔJ is the change in angular momentum of the atom. In contrast, a magnetic dipole (M1) transition satisfies the selection rule: '$\Delta J = \pm 1, 0$; there is no change in parity'. These concepts and the associated selection rules are essentially kinematical in character; dynamics enters in only in comparing the importance of higher multipoles M2, E3, . . . relative to E1, and of E2, M3, . . . relative to M1.

The concept of a relativistic (or 'hindered') M1 transition is, on the contrary, purely dynamical in character. It is illustrated most simply by reference to the low-lying S states of the hydrogen atom, the 1S and 2S hyperfine doublets. In a non-relativistic description, the wavefunction ψ of such a state has the form $\psi(\boldsymbol{r}) = R(r)\,\chi$ where $\boldsymbol{r}$ is the relative electron–proton coordinate and χ is the electron–proton Pauli spin wavefunction, corresponding to total spin F either 0 or 1. If retardation is

neglected, the matrix element for the transition $\psi_i \to \psi_f$ with emission of a photon of momentum $\boldsymbol{k}$ and polarisation $\boldsymbol{\epsilon}$ is then proportional to $\boldsymbol{\epsilon}^* \times \boldsymbol{k}.\boldsymbol{M}$ where $\boldsymbol{M} = \langle \psi_f | \boldsymbol{\mu} | \psi_i \rangle = \langle R_f | R_i \rangle \langle \chi_f | \boldsymbol{\mu} | \chi_i \rangle$ and $\boldsymbol{\mu}$ is the electron spin magnetic moment. For a transition to the $F=0$ ground state from its $F=1$ hyperfine partner we have $R_f = R_i$ so that $\langle R_f | R_i \rangle = 1$ and $\boldsymbol{M}$ is non-vanishing; this transition, responsible for the famous 21 cm line of hydrogen, is an example of an ordinary (or 'unhindered') M1 transition. However, for the transition of the first spatially excited S state 2s to the ground-state configuration 1s, R_f and R_i are orthogonal so that $\langle R_f | R_i \rangle = 0$ and $\boldsymbol{M}$ vanishes. The M1 transition in question nevertheless takes place, as we shall see, when effects neglected in the extreme non-relativistic description are taken into account. This is the simplest example of a relativistic M1 transition.

More generally, a relativistic M1 transition is a radiative M1 transition between states of the same parity but with orthogonal spatial wavefunctions—a highly forbidden transition in atomic physics. Interest in forbidden transitions was, for a long time, restricted to astrophysics, where collisional de-excitation can be sufficiently slow to allow metastable states to live out their natural life, to then die a natural death with the emission of one (or more) photons. Subsequently, such transitions became of interest for solar physics, in connection with the study of the soft x-ray spectrum of the solar corona. However, the subject really came into its own with the development of atomic beam–foil spectroscopy, associated with the construction of heavy-ion linear accelerators. In 1970 this led to the first direct detection in the laboratory of the *decay* of an atomic state via a forbidden transition, namely $2^3S_1 \to 1^1S_0 + \gamma$ in Ar^{16+} (helium-like argon) and, subsequently, to the measurement of the lifetime of the state. Since then, there has been a host of further measurements of this kind, in both H-like and He-like ions, ranging from sulphur ($Z=16$) all the way to krypton ($Z=36$). This has been accompanied by a corresponding increase in theoretical activity, some of which will be described.

By one of those twists which illustrate the unity of nature and of science, the spectacular discovery of the previously mentioned narrow resonances J/ψ and ψ', with masses of the order of 3·1 GeV and 3·7 GeV respectively, but widths of the order of 0·1 MeV, has led to an interest in both ordinary and relativistic M1 transitions in elementary particle physics. This is because in one of the most popular views of these new particles they are pictured as non-relativistic bound states of a heavy quark 'c' and an antiquark '$\bar{c}$'. For reasons connected with the theory of weak interactions, these quarks are thought to carry a new quantum number, universally called 'charm'. The 'psions' ψ and ψ', both $J=1$ states, are then regarded respectively as ground and excited $^3S_1$ states of a bound $(c\bar{c})$ system, called 'charmonium' in analogy with positronium, the bound (e^+e^-) system. The charmonium model predicts, among other things, the existence of a $^1S_0$ state, the 'hyperfine' partner of ψ, into which ψ and ψ' should decay via ordinary and relativistic M1 transitions respectively. Thus there has emerged an entirely new 'atomic' spectroscopy.

An additional motivation for the joint review of these questions stems from the following. The past three years have witnessed an exciting array of discoveries at the frontier of our knowledge of the submicroscopic world: (i) the confirmation of the speculated existence of 'neutral current' interactions between leptons and hadrons, typified by the observation of, say, $\bar{\nu}_\mu + p \to \bar{\nu}_\mu + p$ at a strength comparable to an ordinary 'charged-current' weak interaction such as $\bar{\nu}_\mu + p \to \mu^+ + n$; (ii) the discovery of the narrow vector meson resonances ψ and ψ' referred to above, and the subsequent discovery of other, closely related, states in the 3–4 GeV range, and (iii) the discovery

of predicted new hadrons, with masses around 2 GeV, which are best interpreted as states carrying a non-zero value of the charm quantum number, the so-called 'charmed particles'. The importance of these developments for the formulation of a fundamental unified theory of the strong, electromagnetic and weak interactions is obvious. Moreover, they have brought with them an unexpected but welcome by-product, to which the present review will hopefully contribute. This is renewed interaction and communication among workers in atomic, nuclear and elementary particle physics—disciplines which have a common origin but which have been, for many years, growing further and further apart. An impressive recent example of this new cohesion is provided by the Ben Lee Memorial Conference, Fermilab, October 1977, which brought together workers in all these areas.

A major factor in this has been the search for parity violation effects in atomic and nuclear physics, which are predicted by unified gauge theories of weak and electromagnetic interactions. The fact that the design of some of the experiments is based on the slow rate of relativistic M1 transitions gives yet another reason for interest in them.

The succeeding sections are organised as follows. In §2 the theory of a relativistic magnetic dipole transition is first given for $2^1S_{1/2}\rightarrow 1^1S_{1/2}+\gamma$ in H-like ions and, with this as background, the history of the development of interest in such transitions in H, He and He-like ions is sketched. In §3 the theory of $2^3S\rightarrow 1^1S_0+\gamma$ in He and He-like ions is developed and the theoretical results are compared with experiment. In §4 the developments in particle physics leading to the charmonium model for the psions are reviewed and in §5 this model is described. In §6 the theory of the radiative decays of bound states of two spin-$\frac{1}{2}$ particles is presented in rather general terms, without commitment to a specific dynamics. In §7 the formalism is applied to the M1 decays of the psions and a brief survey of the present status of the charmonium model is given. Concluding remarks are contained in §8.

2. Theory of M1 transitions in H and H-like ions

As a prototype example of a relativistic M1 transition we shall consider, in some detail, the decay $2^2S_{1/2}\rightarrow 1^2S_{1/2}+\gamma$ in H or H-like ions. With this as background it will be easier to describe the history of the development of interest in these transitions and analogous ones in He and He-like ions. We first develop the theory in the Schrödinger–Pauli approximation, with the electron and its interaction with the electromagnetic field treated non-relativistically, and then repeat the calculation using the Dirac one-electron theory; in both cases we shall neglect nuclear recoil and hyperfine effects. Our units are such that $\hbar=c=1$ and $\alpha=e^2/4\pi\simeq 1/137$ is the fine-structure constant.

2.1. Non-relativistic theory

In the Schrödinger–Pauli (SP) approximation the wavefunction for the electron (mass m, charge $-e$) is a two-component Pauli spinor $\Phi(\boldsymbol{r})$ satisfying:

$$H_{\rm SP}\Phi=W\Phi \qquad H_{\rm SP}\equiv\frac{\boldsymbol{p}^2}{2m}+V(r)+H_{\rm fs} \tag{2.1}$$

where $V(r)=-Z\alpha/r$ for a point nucleus of charge Ze and $H_{\rm fs}$ is the operator describing

fine-structure effects, whose explicit form we shall not need. The interaction of the electron with the quantised transverse radiation field $\boldsymbol{A}_T(\boldsymbol{r})$ is given by:

$$H_{SP}' = e\boldsymbol{p}.\boldsymbol{A}_T/m + e^2\boldsymbol{A}_T^2/2m - \boldsymbol{\mu}.\boldsymbol{H}_T \tag{2.2}$$

where $\boldsymbol{H}_T = \nabla \times \boldsymbol{A}_T$ and $\boldsymbol{\mu} = g(-e/2m)\,\boldsymbol{s}$ is the electron spin magnetic moment, g is the gyromagnetic ratio (≈ 2) and $\boldsymbol{s}$ is the electron spin operator ($\boldsymbol{s} = \boldsymbol{\sigma}/2$, $\boldsymbol{\sigma}$ the Pauli spin vector).

The amplitude for the transition from an initial state Φ_i to a final state Φ_f with emission of a photon of polarisation $\boldsymbol{\epsilon}$ and momentum $\boldsymbol{k}$ is then given, to lowest order in the charge e entering in (2.2), by $\mathcal{M} = eM_{SP}/\sqrt{2k}$ where:

$$M_{SP} = \left\langle \Phi_f \left| \left(\frac{\boldsymbol{p}}{m} - i\frac{g}{2}\,\boldsymbol{s} \times \frac{\boldsymbol{k}}{m} \right).\boldsymbol{\epsilon}^* \exp(-i\boldsymbol{k}.\boldsymbol{r}) \right| \Phi_i \right\rangle. \tag{2.3}$$

Let us restrict our attention to even-parity states with $j = \frac{1}{2}$, i.e. to S states, for which the general form of Φ is $R(r)\,\chi/\sqrt{4\pi}$ where $R(r)$ is a radial wavefunction and χ is a constant Pauli spinor. Since the vector $\langle R_f | \boldsymbol{p} \exp(-i\boldsymbol{k}.\boldsymbol{r}) | R_i \rangle$ is by rotational invariance proportional to $\boldsymbol{k}$ and $\boldsymbol{k}.\boldsymbol{\epsilon} = 0$, the orbital term in (2.3), involving $\boldsymbol{p}/m$, vanishes and only the spin term remains. Putting $g = 2$ and integrating over angles in (2.3), we find:

$$M_{SP} = i\Sigma_{fi} T_{SP} \tag{2.4}$$

where

$$\Sigma_{fi} = \left\langle \chi_f \left| \boldsymbol{\sigma} \times \frac{\boldsymbol{k}}{m}.\boldsymbol{\epsilon}^* \right| \chi_i \right\rangle \tag{2.5}$$

and

$$T_{SP} = -\tfrac{1}{2}\langle R_f | j_0(kr) | R_i \rangle \tag{2.6}$$

with $j_0(\rho) = \sin\rho/\rho$, the spherical Bessel function of order zero. Since R_f and R_i are eigenfunctions of H_{SP} belonging to different eigenvalues, but with the same orbital angular momentum, we have:

$$\langle R_f | R_i \rangle = 0. \tag{2.7}$$

This is true regardless of the form of H_{fs} as long as H_{SP} is Hermitian. Now $k \sim (\alpha Z)^2\, m$, $\langle r \rangle \sim (\alpha Z m)^{-1}$ so that $\langle kr \rangle \sim \alpha Z$. Using (2.7) and the expansion $j_0(\rho) = 1 - \rho^2/6 + \ldots$ in (2.6) we see that, to leading order in αZ, $M_{SP} \to M_{SP}^{(0)}$ where:

$$M_{SP}^{(0)} = i\Sigma_{fi}^{(0)} T_{SP}^{(0)} \tag{2.8}$$

with

$$T_{SP}^{(0)} = \frac{1}{12}\langle R_f^{(0)} | (k^{(0)}r)^2 | R_i^{(0)} \rangle. \tag{2.9}$$

Here the superscript zero indicates that the R and k have been replaced by their extreme non-relativistic values, i.e. $R^{(0)}$ is an eigenfunction of $H_0 = \boldsymbol{p}^2/2m + V(r)$ with H_{fs} neglected and $k^{(0)}$ denotes the difference of the eigenvalues of H_0.

Since $\Sigma_{fi}^{(0)} \sim (\alpha Z)^2$ and $T_{SP}^{(0)} \sim (\alpha Z)^2$ we see from (2.8) and (2.9) that:

$$M_{SP} \sim (\alpha Z)^4 \tag{2.10}$$

and that unless Z is extremely large, the transition in question is highly suppressed compared to an E1 transition, for which the amplitude is $M_{SP} = \langle \Phi_f | \boldsymbol{\epsilon}^*.\boldsymbol{p}/m | \Phi_i \rangle \sim \alpha Z$. Note that an extra factor of $(\alpha Z)^2$ in (2.10) arose directly from the orthogonality of R_f and R_i.

Our first concrete example of a relativistic M1 transition shows that the adjective 'relativistic' is somewhat misleading, because the amplitude is found to be non-vanishing even when the electron is treated non-relativistically. One gets a non-zero amplitude as long as the retardation effects represented by inclusion of the higher-order terms contained in the factor $\exp(-i\boldsymbol{k}.\boldsymbol{r})$ are not neglected. Using the expansion

$$\exp(-i\boldsymbol{k}.\boldsymbol{r}) = 1 - i\boldsymbol{k}.\boldsymbol{r} - \frac{(\boldsymbol{k}.\boldsymbol{r})^2}{2!} + \ldots$$

one finds that the first term gives zero by orthogonality, the second by parity, but the third gives (2.9). However, we shall see that the above treatment, although correct as to order of magnitude, yields the wrong answer (it is too small by a factor of 2). To get the right answer, the electron must be treated relativistically—more precisely, *relativistic effects which contribute to the leading term in M are missed if one treats the electron in the Schrödinger–Pauli approximation.* This is *not* because of the approximations involved in (2.1), but rather because the interaction of the electron with the radiation field is not described with sufficient accuracy by (2.2).

The reader may have wondered why we compared the magnitude of the relativistic M1 amplitude (2.10) to that of an E1 amplitude, rather than that for an 'ordinary' M1 amplitude, which is nominally expected to be of order v/c relative to an E1 amplitude, i.e. of order $(\alpha Z)^2$ in an atom. The answer is that the nominal estimate is wrong for an atom and ordinary transitions are also strongly suppressed. To see this, we note that the usual form for an M1 amplitude, obtainable from (2.3) by replacing $\boldsymbol{s}\exp(-i\boldsymbol{k}.\boldsymbol{r})$ by $\boldsymbol{s}$, $\boldsymbol{p}\exp(-i\boldsymbol{k}.\boldsymbol{r})$ by $\boldsymbol{p}(1-i\boldsymbol{k}.\boldsymbol{r})$ and separating $\boldsymbol{\epsilon}.\boldsymbol{p}\,\boldsymbol{k}.\boldsymbol{r}$ into M1 and E2 terms (see, for example, Bethe and Salpeter 1957) is:

$$M = i\frac{\boldsymbol{\epsilon}^* \times \boldsymbol{k}}{2m}.\boldsymbol{M}_{fi} \qquad \boldsymbol{M}_{fi} = \langle \Phi_f | \boldsymbol{l} + 2\boldsymbol{s} | \Phi_i \rangle \tag{2.11}$$

where $\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p}$. If we consider a transition within the same doublet, e.g. $^2P_{3/2} \rightarrow {}^2P_{1/2} + \gamma$, then $|\boldsymbol{M}_{fi}| \sim 1$ but now $k/m \sim (\alpha Z)^4$ so that $M \sim (\alpha Z)^4$ as in (2.10). (Because the phase space is proportional to k, the rate for such an 'unhindered' transition is in fact much smaller than that for a 'hindered' one!) For a transition between different multiplets k/m can be of the order of $(\alpha Z)^2$ but now $\boldsymbol{M}_{fi}$ vanishes if the extreme non-relativistic wavefunctions are used, so that $\boldsymbol{M}_{fi}$ is very small.

For later use let us note that for a many-electron atom $\boldsymbol{s}$ and $\boldsymbol{l}$ are simply replaced by the total spin $\boldsymbol{S}$ and orbital angular momentum $\boldsymbol{L}$:

$$\boldsymbol{M}_{fi} = \langle \Phi_f | \boldsymbol{L} + 2\boldsymbol{S} | \Phi_i \rangle \tag{2.12}$$

and the situation remains unchanged. All this is a consequence of the fact that in an atom, unlike a nucleus, spin-dependent interactions are small and the dynamic constituents all have the same charge and magnetic moment.

2.2. Relativistic theory

In the Dirac one-electron theory, the electron wavefunction ψ is a four-component Dirac spinor, satisfying:

$$H_D\psi = E\psi \qquad H_D = \boldsymbol{\alpha}.\boldsymbol{p} + \beta m + V(r) \tag{2.13}$$

where $\boldsymbol{\alpha}$ and β are the usual Dirac matrices, and the interaction with the transverse

radiation field is:

$$H_D' = e\boldsymbol{\alpha}.\boldsymbol{A}_T(\boldsymbol{r}). \tag{2.14}$$

The matrix element for a transition $\psi_I \to \psi_F$, with emission of one photon, is now $\mathcal{M} = eM_D/\sqrt{2k}$ where:

$$M_D = \langle \psi_F | \boldsymbol{\alpha}.\boldsymbol{\epsilon}^* \exp(-i\boldsymbol{k}.\boldsymbol{r}) | \psi_I \rangle. \tag{2.15}$$

For the decay of interest, viz $2^2S_{1/2} \to 1^2S_{1/2} + \gamma$, M_D may be evaluated exactly when V is the Coulomb potential (Johnson 1972). However, it is more instructive, and for our purposes more useful, to find the leading term in an expansion in powers of αZ of the right-hand side of (2.15) without, moreover, specifying the precise form of V.

The general form of the Dirac wavefunction ψ for a $^2S_{1/2}$ state is:

$$\psi = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} g(r) & \chi \\ -if(r)\,\boldsymbol{\sigma}.\hat{\boldsymbol{r}}\chi \end{pmatrix} \qquad f = \frac{g'(r)}{E+m-V}$$

where χ is a Pauli spinor. Then M_D reduces to

$$M_D = i\Sigma_{fi} T_D \tag{2.16}$$

where Σ_{fi} is still defined by (2.5) but:

$$T_D = (m/k)[\langle g_F | j_1(kr) | f_I \rangle + \langle f_F | j_1(kr) | g_I \rangle] \tag{2.17}$$

with j_1 the spherical Bessel function of order one.

If in (2.17) one replaces $g(r)$, $f(r)$ and $j_1(kr)/k$ by their non-relativistic limits:

$$g^{(0)} = R(r) \qquad f^{(0)} = R'(r)/2m \qquad (j_1/k)^{(0)} = \tfrac{1}{3}r \tag{2.18}$$

where $R(r)$ is the completely non-relativistic radial wavefunction, then on integration by parts one finds, as expected:

$$T_D \to \text{constant} \times \langle R_F | R_I \rangle = 0$$

which just corresponds to the characterisation of the transition as relativistic. The departure of g, f and j_1/k from (2.18) leads to three distinct types of contributions to $T_D^{(0)}$, the lowest-order term in an expansion of T_D in powers of αZ. To find these we note first that if (j_1/k) is replaced by $\frac{1}{3}r$ and f is approximated by $g'(r)/2m$, then after an integration by parts one gets a term $-\langle g_F | g_I \rangle/2$, which does not vanish. On the contrary, $\langle \psi_F | \psi_I \rangle = 0$ implies that $\langle g_F | g_I \rangle = -\langle f_F | f_I \rangle \approx -\langle g_F' | g_I' \rangle/4m^2 = -\langle g_F | \boldsymbol{p}^2/4m^2 | g_I \rangle$, so that we find a 'large component' correction:

$$T_D{}^g = \langle R_F | \boldsymbol{p}^2/8m^2 | R_I \rangle. \tag{2.19}$$

Similarly, if one takes into account the difference $f-(g'/2m)$, but neglects $g-R$ and $(j_1/k)-\frac{1}{3}r$ one gets a 'small component' correction:

$$T_D{}^f = \langle R_F | 5\boldsymbol{p}^2/24m^2 - rV'(r)/6m | R_I \rangle. \tag{2.20}$$

Finally, the second term in the expansion $j_1(\rho) = \frac{1}{3}\rho - \rho^3/30 + \ldots$ yields a 'retardation term':

$$T_D{}^{\text{ret}} = \langle R_F | k^2r^2/12 | R_I \rangle \tag{2.21}$$

where k is now the non-relativistic energy difference $W_I - W_F$. The sum of (2.19), (2.20) and (2.21) is the sought-after quantity:

$$T_D{}^{(0)} = \langle R_F | \boldsymbol{p}^2/3m^2 - rV'(r)/6m + k^2r^2/12 | R_I \rangle. \tag{2.22}$$

The form (2.22) will be useful for later comparison with the $2^3S_1 \rightarrow 1^1S_0 + \gamma$ amplitude in He-like ions. For the hydrogenic case it is convenient to use the identity:

$$\langle R_F | k^2 r^2 | R_I \rangle = \langle R_F | [\boldsymbol{p}^2/2m, [\boldsymbol{p}^2/2m, r^2]] | R_I \rangle = \langle R_F | -2\boldsymbol{p}^2/m^2 + 2rV'/m | R_I \rangle$$

to rewrite $T_D^{(0)}$ in the simpler form:

$$T_D^{(0)} = \langle R_F | \boldsymbol{p}^2/6m^2 | R_I \rangle \tag{2.23}$$

which does not depend explicitly on the form of V.

It should be noted that T_D^{ret} coincides with $T_{SP}^{(0)}$, the leading term (2.9) found in the non-relativistic theory based on (2.1) and (2.2). Thus the sum:

$$T_D^{\text{rel}} = T_D^f + T_D^g = \langle R_F | \boldsymbol{p}^2/3m^2 - rV'(r)/6m | R_I \rangle \tag{2.24}$$

represents the effect of treating the interaction of the electron with the radiation field relativistically. For a Coulomb potential $rV' = -V$ so that one finds, using the relation $\langle R_F | V | R_I \rangle = -\langle R_F | \boldsymbol{p}^2/2m | R_I \rangle$, that:

$$T_D^{\text{rel}} = \tfrac{3}{2} T_D^{(0)} \qquad T_D^{\text{ret}} = -\tfrac{1}{2} T_D^{(0)}. \tag{2.25}$$

It follows that neglect of the specific relativistic effects contained in (2.24) would lead to a rate which is too small by a factor of $\frac{1}{4}$ whereas neglect of the retardation effect gives a rate too large by a factor of 9/4; the literature contains examples of both of these possible types of neglect.

The decay rate is given by:

$$R_{\text{H-like}} = 2\pi \int \frac{d\boldsymbol{k}}{(2\pi)^3} \delta(E_f + k - E_i) \sum_{\text{pol., spin}} |\mathcal{M}|^2$$

which yields:

$$R_{\text{H-like}} = \frac{4\alpha k^3}{m^2} |T_D|^2. \tag{2.26}$$

To lowest order in αZ, $k \rightarrow k^{(0)} = \frac{3}{8}(\alpha Z)^2 m$ and $T_D \rightarrow T_D^{(0)} = (4\sqrt{2}/81)(\alpha Z)^2$ so that:

$$R_{\text{H-like}}^{(0)}(2^2S_{1/2} \rightarrow 1^2S_{1/2} + \gamma) = \frac{1}{972} \alpha(\alpha Z)^{10} m. \tag{2.27}$$

For $Z = 1$, (2.27) yields the decay rate R_H of the 2s state of hydrogen:

$$R_H = 2{\cdot}50 \times 10^{-6}\ \text{s}^{-1} \tag{2.28}$$

so that the lifetime $\tau_H = R_H^{-1}$ is 4×10^5 s (about five days). Note that the rate scales with an extraordinarily large power of Z: $R_{\text{H-like}} = Z^{10} R_H$, so that already for $Z \sim 25$ the rate of the transition is comparable to that of an E1 transition in hydrogen and use of the adjective 'forbidden' should be avoided (if not forbidden).

2.3. How do metastable S states decay? A brief history

The first extensive discussion of the lifetimes of the excited states of the hydrogen atom appears to have been given by H A Bethe, in his classic article in the *Handbuch der Physik* (Bethe 1933). Referring to the question of whether each state of the degenerate fine-structure doublet with $l = j - \frac{1}{2}$ and $l = j + \frac{1}{2}$ decays separately or whether instead one should use some mean lifetime, Bethe writes: 'On this question

there have been so many mutually contradictory assertions that we consider it necessary to discuss it thoroughly' [author's translation]. His main focus is, however, on the effect of external fields in mixing the 2s and 2p states. After characterising the first excited S state as metastable, with an 'infinite lifetime', he remarks, in a footnote: 'Speaking more precisely, the lifetime is finite as soon as the exact relativistic wavefunctions are used. But the lifetimes of the 2s-state is several months, if it can only disintegrate by a radiative process' [author's translation].

The reference to relativistic wavefunctions indicates that Bethe had in mind only the one-photon decay, $2s \rightarrow 1s + \gamma$. The possibility of two-photon decay of an atomic system, within the framework of Dirac's radiation theory, had been stressed earlier by Goeppert-Mayer (1929, 1931), but she had not estimated the magnitude of the rate for any such decay†. The subject was taken up again by Breit and Teller (1940), who were motivated by questions of astrophysical interest. Much of their pioneering paper is concerned with an analysis of the effect of collisions with electrons on the lifetime of the 2s state of hydrogen, but it also contains the first detailed discussion of the one-photon and two-photon radiative decays of the metastable states of hydrogen ($2^2S_{1/2}$) and helium (2^1S_0 and 2^3S_1), to the ground state, with quantitative numerical results for $2^2S_{1/2} \rightarrow 1^2S_{1/2} + n\gamma$ in H, and order-of-magnitude estimates for $2^1S_0 \rightarrow 1^1S_0 + n\gamma$ and $2^3S_1 \rightarrow 1^1S_0 + n\gamma$ in He ($n = 1, 2$). For hydrogen, Breit and Teller found:

$$6{\cdot}5\ \mathrm{s}^{-1} < R_{\mathrm{H}}(2^2S_{1/2} \rightarrow 1^2S_{1/2} + 2\gamma) < 8{\cdot}4\ \mathrm{s}^{-1} \qquad (a)$$

and

$$R_{\mathrm{H}}(2^2S_{1/2} \rightarrow 1^2S_{1/2} + \gamma) = \frac{1}{432}\,\alpha^{11} m = 5{\cdot}6 \times 10^{-6}\ \mathrm{s}^{-1}. \qquad (b)$$

For the two-photon decays in He they estimated:

$$R_{\mathrm{He}}(2^1S_0 \rightarrow 1^1S_0 + 2\gamma) \sim R_{\mathrm{H}}(2^2S_{1/2} \rightarrow 1^2S_{1/2} + 2\gamma) \qquad (c)$$

$$R_{\mathrm{He}}(2^3S_1 \rightarrow 1^1S_0 + 2\gamma) \sim 10^{-6}\, R_{\mathrm{H}}(2^2S_{1/2} \rightarrow 1^2S_{1/2} + 2\gamma). \qquad (d)$$

The one-photon decay $2^1S_0 \rightarrow 1^1S_0 + \gamma$ is strictly forbidden by the $J = 0 \not\rightarrow J = 0$ selection rule; for the one-photon decay of the 2^3S_1 state their result may be stated in the form:

$$R_{\mathrm{He}}(2^3S_1 \rightarrow 1^1S_0 + \gamma) \sim 10^{-24}\, R \qquad (e)$$

where R denotes the rate expected for an E1 decay with unit oscillator strength.

The results (a) to (d) have stood the test of time relatively well. The striking result (a), usually quoted as $\tau \sim \frac{1}{7}$ s, showed that the lifetime of the hydrogenic 2s state is much shorter than was previously suspected; a modern value for τ^{-1} is $8{\cdot}23\ \mathrm{s}^{-1}$ (Klarsfeld 1969), within the bounds of (a). The lifetime for the one-photon decay corresponding to (b), about 2 days, is also considerably smaller than the 'several months' of Bethe; the result (b) exceeds the correct value (2.28) by a factor 9/4, apparently because Breit and Teller prematurely replaced $\exp(-i\boldsymbol{k}.\boldsymbol{r})$ by $1 - i\boldsymbol{k}.\boldsymbol{r}$ in their evaluation of (2.15), thereby missing the retardation term exhibited in (2.21). With regard to the two-photon decays in He, the presently accepted values are, for the 2^1S_0 state (Drake *et al* 1969, Jacobs 1971):

$$R_{\mathrm{He}}(2^1S_0 \rightarrow 1^1S_0 + 2\gamma) \approx 51\ \mathrm{s}^{-1} \qquad (2.29)$$

† The contrary was implied in an earlier report (Sucher 1976) on the basis of an incorrect inference drawn from a reading of only secondary sources.

with which the strictly order-of-magnitude estimate (*c*) is not inconsistent, and for the 2^3S_1 state (Drake *et al* 1969, Bély and Faucher 1969):

$$R_{\mathrm{He}}(2^3S_0 \rightarrow 1^1S_0 + 2\gamma) \approx 4 \times 10^{-9}\ \mathrm{s}^{-1} \tag{2.30}$$

which differs from (*d*) by a few powers of ten.

With regard to (*e*) for the M1 decay, Breit and Teller state: 'This effect is so small that one does not need a more accurate estimate'. Indeed, if one takes $R < 10^9\ \mathrm{s}^{-1}$ (the maximum decay rate in the H atom, for $2\mathrm{p} \rightarrow 1\mathrm{s} + \gamma$, has the value $6 \times 10^8\ \mathrm{s}^{-1}$), one would infer from (*e*) that $R_{\mathrm{He}}(2^3S_1 \rightarrow 1^1S_0 + \gamma) < 10^{-15}\ \mathrm{s}^{-1}$ and hence that the 1γ decay rate of this state is down by a factor of 10^{-5} or less from the 2γ decay, even if the modern value (2.30) for the latter rate is used. However, the estimate (*e*) is incorrect. As we shall see, the 1γ decay is in fact the *dominant* decay mode of the 2^3S_1 state, with a rate which is larger than the 2γ rate by a factor of 10^5!

The reader may be curious about the source of the estimate (*e*). It was the result of a strange oversight rather than an error in calculation. In their study of the M1 decay of 2^3S_1 in He, in contrast to their study of the M1 decay of $2^1S_{1/2}$ in H, Breit and Teller consider only the usual form of an M1 amplitude, already mentioned in §2.1, $M = \mathrm{i}\boldsymbol{\epsilon}^* \times \boldsymbol{k} . \boldsymbol{M}_{\mathrm{fi}}/2m$ with $\boldsymbol{M}_{\mathrm{fi}}$ given by (2.12). Since $\boldsymbol{L} + 2\boldsymbol{S} = \boldsymbol{J} + \boldsymbol{S}$ and $J_{\mathrm{f}} \neq J_{\mathrm{i}}$ one may write:

$$\boldsymbol{M}_{\mathrm{fi}} = \langle \Phi_{\mathrm{f}} | \boldsymbol{S} | \Phi_{\mathrm{i}} \rangle . \tag{2.31}$$

It is easy to see that $\boldsymbol{M}_{\mathrm{fi}}$ is extremely small. The general form of the $J = 1$ state Φ_{i} and the $J = 0$ state Φ_{f} is, allowing for the mixing effects of the fine-structure operator H_{fs}:

$$\begin{aligned} \Phi_{\mathrm{i}} &= a_0 |{}^3S_1\rangle + a_1 |{}^3P_1\rangle + a_2 |{}^3D_1\rangle + a_3 |{}^1P_1\rangle \\ \Phi_{\mathrm{f}} &= b_0 |{}^1S_0\rangle + b_1 |{}^3P_0\rangle \end{aligned} \tag{2.32}$$

where a_0 and b_0 are of the order of unity, and the other a_1 and b_1 are of the order of $\langle H_{\mathrm{fs}} \rangle / \alpha^2 m \sim \alpha^2$. It follows that $M_{\mathrm{fi}} \sim b_1^* a_1 \sim \alpha^4$ from *this* mechanism and hence that one gets a contribution to the decay amplitude which is of the order of α^6 compared to an E1 amplitude of the order of α; hence, the corresponding rate would appear to be down by at least a factor of $\alpha^{10} \sim 4 \times 10^{-22}$, which accounts for the majestically small coefficient in front of the E1 rate R in (*e*). However, in this estimate the baby has been thrown out with the bathwater. There is a much larger contribution coming from retardation and relativistic effects—the latter are entirely analogous to those which Breit and Teller themselves were the very first to consider in detail for the 2s decay of H, in the last few pages of their paper, but which have already been neglected in the expression (2.31).

We have gone to some length in describing the results of Breit and Teller because their work of 1940 was not only the first but also the final word on this subject for almost *thirty years*, until 1969.

In 1969, the Sun enters our story. It was then noted by Gabriel and Jordan (1969a) that certain lines seen in the soft x-ray spectrum of the solar corona correspond in wavelength to the transition:

$$2^3S_1 \rightarrow 1^1S_0 + \gamma \tag{2.33}$$

in certain *He-like ions*: C V, O VII, Ne IX, Mg XI and Si XIII, and they suggested that these transitions were in fact responsible for the observed lines. However, the intensities of these lines would be extremely small if the estimate (*e*) of Breit and

Teller for helium were correct, even if it were scaled with a high power of Z. Griem (1969) then showed that the Gabriel–Jordan interpretation of these lines is plausible by pointing out that Breit and Teller vastly underestimated the rate for the transition in question (for helium). His estimate of the M1 decay rate (patterned after the original Breit–Teller calculation for hydrogen) indicated that the rate is sufficiently large, compared to collisional de-excitation rates, to explain the strong emission observed, and to make 2γ decay negligible for $Z \lesssim 20$.

A correct order-of-magnitude estimate of the M1 decay rate of the 2^3S_1 state in He was also made at about the same time by G Feinberg (1969, private communication) who had independently noted that the dominant effect had been overlooked; the motivation here was provided by astrophysical speculations involving the extent of ionisation of He in the Universe (see Rees and Sciama 1969).

This completes our survey of the early theoretical work. We should add that Gabriel and Jordan (1969b) also developed a scheme to determine electron densities in the solar corona, based on the observed intensities of the lines in question and on theoretical decay rates. It thus became of interest to carry out more refined calculations of these rates. Such calculations soon became of even greater interest with the advent of accurate experimental results from beam–foil spectroscopy for decays with $Z \geqslant 16$, beginning in 1970 with the observation of (2.33) in Ar^{16+} and with measurement of the rate by Marrus and Schmieder (1970a,b). One might say that with these measurements the subject had finally come down to earth.

3. $2^3S_1 \rightarrow 1^1S_0 + \gamma$ in He and He-like ions: experiment and theory

3.1. Experimental developments in the 1970s

'In spite of the fact that the hydrogen and helium spectra can probably qualify among the most studied problems in the history of physics, there is relatively little information concerning the forbidden decays, i.e. those decays which proceed by other than an allowed electric dipole transition'.

This statement, the opening sentence in a paper of Marrus and Schmieder (1972), is happily no longer descriptive of the situation, at least as far as the M1 decays of He-like ions for large Z are concerned. The lifetime for the 2^3S_1 state has now been measured for many such ions, ranging from $Z=16$ (sulphur) to $Z=36$ (krypton). All these measurements have been carried out by the atomic beam–foil method in which ions, emerging from a heavy-ion linear accelerator, are sent through thin foils, thereby stripping them of still more electrons and creating a beam of He-like ions, some of which are in the 2^3S_1 state. Measurement of the intensity of the photons from the M1 decay of this state as a function of the distance from the foil then determines the lifetime. The most recent results of such measurements for helium-like sulphur, chlorine (Bednar *et al* 1975), argon (H Gould and R Marrus 1976 private communication), titanium (Gould *et al* 1973), vanadium, iron (Gould *et al* 1974), and krypton (Gould and Marrus 1976) will be displayed later, after a discussion of the theory of these decays.

It should be noted that the lifetimes range from $\sim 10^{-6}$ s for $Z=16$ to $\sim 10^{-10}$ s for $Z=36$, over more than three orders of magnitude. This corresponds to the fact that for large Z, the rate goes as Z^{10} and $(36/16)^{10} \sim 3 \times 10^3$. These values of Z are near the limits of what can be achieved by these methods: for much larger Z one gets too close to the foil, and for $Z<16$ one begins to run out of real estate. (For $v/c \sim 0{\cdot}1$,

a decay length is 30 m when $\tau = 10^{-6}$ s.) A review of experimental techniques has been given by Marrus (1973).

It would be a pity if one could not add the first member of the helium isoelectronic sequence, ordinary helium, to the list of atoms for which the lifetime is measured. The long lifetime ($\sim 10^4$ s) makes the detection of this decay exceptionally difficult. The 626 Å radiation expected from the transition $2^3S_1 \rightarrow 1^1S_0 + \gamma$ in He I was first seen in the afterglow from a helium radiofrequency discharge by Moos and Woodworth (1973). The rather large error in the initial determination of the rate in this very difficult experiment was subsequently greatly reduced, with the result that $\tau^{-1} = 1{\cdot}10 \times 10^{-4}$ s^{-1}, with a 2σ uncertainty of $\pm 30\%$ (Woodworth and Moos 1975). Note that this rate is bigger than that for $2^2S_{1/2} \rightarrow 1^2S_{1/2} + \gamma$ in H, given by (2.28).

Because of the role played by the solar corona and the fact that helium itself was first discovered in the Sun, it seems only poetic justice that we can indeed add He I to our list. This extends the range of the lifetimes in question to over fourteen orders of magnitude and greatly increases the severity of a comparison of experiment and the theory, to which we now turn.

3.2. Theory of the $2^3S_1 \rightarrow 1^1S_0 + \gamma$ decay

3.2.1. Relativistic effects in many-electron atoms. The extension of the relativistic theory of the $2^2S_{1/2} \rightarrow 1^2S_{1/2} + \gamma$ in H and H-like ions, described in §2.2, to the analogous decay mode $2^3S_1 \rightarrow 1^1S_0 + \gamma$ in He and He-like ions is made complicated by the necessity of including the effects of the electron–electron interaction. Moreover, as the hydrogen example shows, one must avoid making non-relativistic approximations prematurely.

A traditional approach to the calculation of relativistic effects in atoms with more than one electron is based on the use of the Dirac–Breit Hamiltonian:

$$H_{\mathrm{DB}} = \sum_{i=1}^{N} (H_{\mathrm{D}}(\boldsymbol{p}_i) + U(i)) + \sum_{i<j} U_{ij} \tag{3.1}$$

where

$$H_{\mathrm{D}}(\boldsymbol{p}) = \boldsymbol{\alpha}.\boldsymbol{p} + \beta m. \tag{3.2}$$

$U(i)$ denotes the interaction of the electron 'i' with external fields, and:

$$U_{ij} = V_{ij} + B_{ij} \tag{3.3}$$

with

$$V_{ij} = \alpha/r_{ij} \qquad B_{ij} = -\alpha(\boldsymbol{\alpha}_i.\boldsymbol{\alpha}_j + \boldsymbol{\alpha}_i.\hat{\boldsymbol{r}}_{ij}\boldsymbol{\alpha}_j.\hat{\boldsymbol{r}}_{ij})/2r_{ij}. \tag{3.4}$$

Here B_{ij} is the Breit operator which corresponds physically to retardation corrections to the Coulomb interaction V_{ij} as well as spin-dependent magnetic interactions between the electrons; in QED it arises from the exchange of a single transverse photon between the electrons 'i' and 'j'.

However, the use of (3.1) as a starting point is fraught with danger, for two reasons. The first, which is well known, is that if B_{ij} is used in higher orders of perturbation theory, one gets spuriously large contributions to level shifts (see, e.g., Bethe and Salpeter 1957). The second, which has also been known for a long time but appears to have been only weakly transmitted to succeeding generations, is that (3.1) *is unlikely to have bound states at all, even if the B_{ij} terms are omitted*†. The

† Brown and Ravenhall (1951) make this point for He, with U_{12} given by (3.3).

reason is that the normalisable eigenfunctions of the first sum in (3.1), which are products of one-electron Dirac bound-state wavefunctions, lie embedded in a continuum of non-normalisable states involving one or more electrons with negative-energy wavefunctions, so that each point of the zero-order spectrum is non-denumerably degenerate. (Further discussion of this point is given in Sucher (1977); see also Armstrong (1978).)

A consistent treatment of relativistic effects in which such difficulties do not occur can be based on the four-dimensional Bethe–Salpeter equation (Bethe and Salpeter 1951) suitably generalised to take into account interaction of the electrons with an external field (Araki 1957, Sucher 1958a). Such an approach was first used to find the corrections of order α^3 Ryd to the ionisation energy of He and He-like ions (Sucher 1958a,b) and has been used more recently to find the fine-structure splitting of excited states of He to order α^4 Ryd (Douglas and Kroll 1974).

There is, however, another method, which is not only more transparent and simpler conceptually but also simpler from a practical point of view (at least for the problem at hand). Moreover, it generalises immediately to the case of atoms with more than two electrons. This approach (Feinberg and Sucher 1971) starts directly with the Hamiltonian H of quantum electrodynamics, in Coulomb gauge, with an interaction with a static external field $A_{\text{ext}}{}^{\mu}(\boldsymbol{x})$ included in H. In the Schrödinger picture H is given by:

$$H = H_{\text{free}} + H_{\text{ext}} + H_{\text{C}} + H_{\text{T}} \tag{3.5}$$

where H_{free} is the sum of the Hamiltonians of the free Dirac field $\psi_{\text{D}}(\boldsymbol{x})$ and transverse radiation field $A_{\text{T}}(\boldsymbol{x})$ and

$$H_{\text{ext}} = \int j_{\mu}(\boldsymbol{x})\, A_{\text{ext}}{}^{\mu}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \qquad H_{\text{C}} = \tfrac{1}{2} \int\int \frac{j^0(\boldsymbol{x})\, j^0(\boldsymbol{x}')\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{x}'}{4\pi |\boldsymbol{x}-\boldsymbol{x}'|} \tag{3.6}$$

$$H_{\text{T}} = -\int \boldsymbol{j}(\boldsymbol{x}) \cdot \boldsymbol{A}_{\text{T}}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

with $j^{\mu}(\boldsymbol{x}) = -e{:}\bar{\psi}_{\text{D}}(\boldsymbol{x})\, \gamma^{\mu} \psi_{\text{D}}(\boldsymbol{x})$: the electromagnetic current. The method is based on the simple observation that in atoms those effects of the Coulomb interaction H_{C} which involve the virtual creation or destruction of electron–positron pairs lead to level shifts which are of the order of α^3 Ryd or higher. Apart from powers of Z the same holds for the pair effects contained in H_{ext}. Thus one may treat these as perturbations along with H_{T}, which describes the emission or absorption of real or virtual photons. To be precise, let us expand $\psi_{\text{D}}(\boldsymbol{x})$ in the standard way:

$$\psi_{\text{D}}(\boldsymbol{x}) = (2\pi)^{-3/2} \sum_{r=1}^{2} \int \mathrm{d}\boldsymbol{p} \left(\frac{m}{E(\boldsymbol{p})}\right)^{1/2} \times [a_r(\boldsymbol{p})\, u_r(\boldsymbol{p}) \exp(\mathrm{i}\boldsymbol{p}.\boldsymbol{x}) + b_r{}^{+}(\boldsymbol{p})\, v_r(\boldsymbol{p}) \exp(-\mathrm{i}\boldsymbol{p}.\boldsymbol{x})] \tag{3.7}$$

where $u_r(\boldsymbol{p})$ is a positive-energy spinor, $a_r(\boldsymbol{p})$ destroys an electron of momentum $\boldsymbol{p}$, etc. The 'no-pair' part $H_{\text{C}}{}^{\text{np}}$ of H_{C} contains only terms of the form $a^{+}a^{+}aa$ or $b^{+}b^{+}bb$ or $a^{+}b^{+}ab$, while $H_{\text{ext}}{}^{\text{np}}$ contains only $a^{+}a$ or $b^{+}b$ terms. Terms such as

$$a^{+}(\boldsymbol{p}_1)\, b^{+}(\boldsymbol{q}_1)\, a^{+}(\boldsymbol{p}_2')\, a(\boldsymbol{p}_2)$$

in H_{C} or $a^{+}(\boldsymbol{p}_1)\, b^{+}(\boldsymbol{q}_1)$ in H_{ext}, which may be represented schematically as shown in figure 1(*a*) and (*b*), are included as a perturbation H' along with H_{T}. Thus our decomposition is:

$$H = H^{\text{np}} + H' \tag{3.8}$$

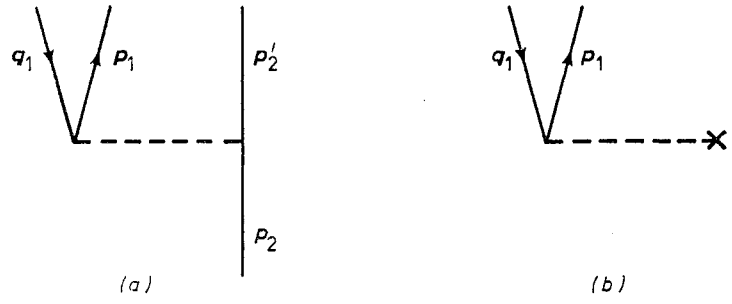

Figure 1. Time-ordered Feynman-like diagrams symbolising virtual-pair production by (*a*) an electron and (*b*) an external static electromagnetic field. The full lines indicate electrons if the arrow points up, positrons if down; the broken lines indicate an 'instantaneous' interaction and the cross the action of the external field.

where:

$$H^{\mathrm{np}} = H_{\mathrm{free}} + H_{\mathrm{ext}}{}^{\mathrm{np}} + H_{\mathrm{C}}{}^{\mathrm{np}} \tag{3.9}$$

and

$$H' = H_{\mathrm{ext}}{}^{\mathrm{pair}} + H_{\mathrm{C}}{}^{\mathrm{pair}} + H_{\mathrm{T}} \tag{3.10}$$

with $H_{\mathrm{ext}}{}^{\mathrm{pair}} \equiv H_{\mathrm{ext}} - H_{\mathrm{ext}}{}^{\mathrm{np}}$, $H_{\mathrm{C}}{}^{\mathrm{pair}} \equiv H_{\mathrm{C}} - H^{\mathrm{np}}$; renormalisation counter-terms in H' are suppressed.

3.2.2. No-pair wave equations. Since H^{np} commutes separately with the electrons and positron number operators N_- and N_+, its eigenstates Ψ may be classified according to the values of $N_\pm$ and modified Dirac equations obtained for appropriately defined wavefunctions. Thus in the one-electron sector of Hilbert space with states $\Psi_{1\mathrm{e}}$ satisfying $N_-\Psi_{1\mathrm{e}} = \Psi_{1\mathrm{e}}$ and

$$H^{\mathrm{np}}\Psi_{1\mathrm{e}} = E\Psi_{1\mathrm{e}} \tag{3.11}$$

we find, on taking the matrix element of (3.11) with states $|\boldsymbol{p}, r\rangle = a_r{}^+(\boldsymbol{p})|\mathrm{vac}\rangle$, multiplying by $u_r(\boldsymbol{p})$ and summing on 'r' that the function

$$\psi(\boldsymbol{p}) \equiv \sum_{r=1}^{2} u_r(\boldsymbol{p})\langle \boldsymbol{p}, r|\Psi_{1\mathrm{e}}\rangle \tag{3.12}$$

satisfies the equation:

$$H_+\psi(\boldsymbol{p}) = E\psi(\boldsymbol{p})$$

$$H_+ \equiv E(\boldsymbol{p}) + \Lambda_+(\boldsymbol{p})\, U \Lambda_+(\boldsymbol{p}). \tag{3.13}$$

$\Lambda_+(\boldsymbol{p})$ is the Casimir projection operator:

$$\Lambda_\pm(\boldsymbol{p}) = \frac{E(\boldsymbol{p}) \pm H_{\mathrm{D}}(\boldsymbol{p})}{2E(\boldsymbol{p})}$$

and $U = -e(A_0{}^{\mathrm{ext}} - \boldsymbol{\alpha}.\boldsymbol{A}^{\mathrm{ext}})$ is the interaction with the external field. Note that from the definition (3.12)—or directly from (3.13) itself(!)—one infers:

$$\Lambda_+\psi = \psi. \tag{3.14}$$

In an entirely similar fashion one finds that if $\Psi_{2\mathrm{e}}$ is in the two-electron sector, with $N_-\Psi_{2\mathrm{e}} = 2\Psi_{2\mathrm{e}}$ and

$$H^{\mathrm{np}}\Psi_{2\mathrm{e}} = E\Psi_{2\mathrm{e}} \tag{3.15}$$

then:

$$\psi(p_1, p_2) \equiv \sum_{r_1, r_2} u_{r_1}(p_1)\, u_{r_2}(p_2) \langle p_1 r_1; p_2 r_2 | \Psi_{2e} \rangle \tag{3.16}$$

satisfies

$$H_{++}\psi = E\psi$$

$$H_{++} \equiv E_1 + E_2 + \Lambda_{++} v \Lambda_{++} \tag{3.17}$$

with

$$v = U(1) + U(2) + V_{12} \tag{3.18}$$

the sum of external field and interelectron Coulomb interactions:

$$\Lambda_{++} = \Lambda_+(p_1)\, \Lambda_+(p_2)$$

and

$$\Lambda_+(p_1)\, \psi = \Lambda_+(p_2)\, \psi = \psi. \tag{3.19}$$

Equation (3.17), a kind of 'no-pair Dirac–Coulomb equation', is also a natural outcome of the approach using the Bethe–Salpeter formalism (Sucher 1958a); here we have obtained it without invoking any four-dimensional machinery. It will form the starting point of the calculation of relativistic M1 decay in He-like ions.

The matrix element for a transition from an eigenstate $|i\rangle = |\Psi_i\rangle$ of $H^{\rm np}$ containing no photons to an eigenstate $|f\rangle = |\Psi_f; \boldsymbol{k}, \boldsymbol{\epsilon}\rangle$ with one photon is given by

$$\mathcal{M} = \langle \Psi_f; \boldsymbol{k}, \boldsymbol{\epsilon} | H' + H'(E_i - H^{\rm np} + i\epsilon)^{-1} H' + \ldots | \Psi_i \rangle. \tag{3.20}$$

We will first illustrate the application of the formalism to M1 decays in H-like ions where, in leading order, virtual photons need not be considered. We will then turn to the more complicated case of He-like ions.

3.2.3. H-like ions revisited. To the order of interest the amplitude M for a one-photon transition between states ψ_i and ψ_f satisfying (3.13) with $U \to U(\boldsymbol{r}) = -eA^0(\boldsymbol{r})$, hyperfine effects being neglected, is given by:

$$M = M^{(1)} + M^{(2)} \tag{3.21}$$

where

$$M^{(1)} = \langle \psi_f | \boldsymbol{\alpha}.\boldsymbol{\epsilon}^* \exp(-i\boldsymbol{k}.\boldsymbol{r}) | \psi_i \rangle \tag{3.22}$$

and

$$M^{(2)} \simeq \langle \psi_f | \boldsymbol{\alpha}.\boldsymbol{\epsilon}^* \exp(-i\boldsymbol{k}.\boldsymbol{r})\, \Lambda_-(p)\, \frac{U(\boldsymbol{r})}{2m} + \frac{U(\boldsymbol{r})}{2m}\, \Lambda_-(p)\, \boldsymbol{\alpha}.\boldsymbol{\epsilon}^* \exp(-i\boldsymbol{k}.\boldsymbol{r}) | \psi_i \rangle. \tag{3.23}$$

Here $M^{(1)}$ and $M^{(2)}$ represent the contribution of the first term and second term of (3.20), symbolised by the time-ordered Feynman-like diagrams shown in figure 2; photon and electron kinetic energies have been neglected compared to $2m$ in the expression for $M^{(2)}$. The evaluation of $M^{(1)}$ to the requisite accuracy can be carried out most straightforwardly by using the exact reduction to 'large components' $\psi^{(+)} = (1+\beta)\,\psi/2$ afforded by the constraint (3.14), viz:

$$\psi = (1+\xi)\, \psi^{(+)} \qquad \xi \equiv \frac{\boldsymbol{\alpha}.\boldsymbol{p}}{m + E(p)}. \tag{3.24}$$

Taking due account, as in §2.2 of the non-orthogonality of $\psi_f^{(+)}$ and ψ_i^{+}:

$$\langle \psi_f^{(+)} | \psi_i^{(+)} \rangle \approx -\langle \psi_f^{(+)} | \boldsymbol{p}^2/4m^2 | \psi_i^{(+)} \rangle$$

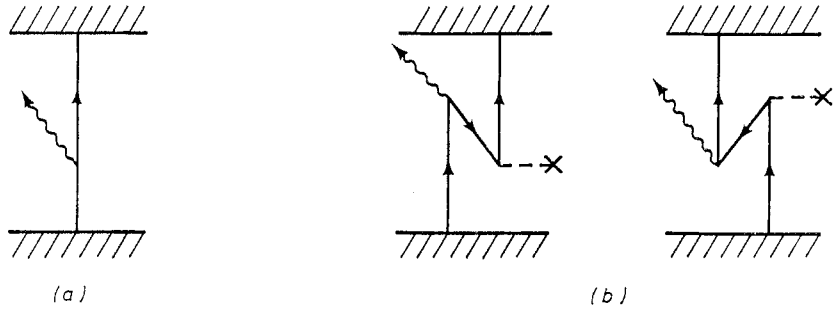

Figure 2. Time-ordered diagrams (*a*) and (*b*), corresponding to the amplitudes $M^{(1)}$ and $M^{(2)}$ of the text, respectively. The shaded portions symbolise integration over bound-state wavefunctions and the wavy line denotes a photon.

and using the approximations

$$\xi \approx \frac{\boldsymbol{\alpha}.\boldsymbol{p}}{2m}\left(1-\frac{\boldsymbol{p}^2}{4m^2}\right)$$

and

$$\exp(-\mathrm{i}\boldsymbol{k}.\boldsymbol{r}) \approx 1-\mathrm{i}\boldsymbol{k}.\boldsymbol{r}-(\boldsymbol{k}.\boldsymbol{r})^2/2$$

one finds†, after some algebra and specialisation to S states, that:

$$M^{(1)} \approx \mathrm{i}\Sigma_{fi}(T^{\mathrm{kin}}+T^{\mathrm{ret}}) \tag{3.25}$$

where

$$T^{\mathrm{kin}} = \langle R_f | \boldsymbol{p}^2/3m^2 | R_i \rangle \tag{3.26}$$

is a 'kinetic correction' arising from the combined effect of the higher-order terms in ξ and the non-orthogonality of the $\psi^{(+)}$ and T^{ret} is the retardation term already encountered in §§2.1 and 2.2:

$$T^{\mathrm{ret}} = \langle R_f | k^2 r^2/12 | R_i \rangle. \tag{3.27}$$

In $M^{(2)}$ the non-relativistic limit for the wavefunctions may be taken immediately ($\psi^{(+)} \rightarrow R\chi$), with the result that, for $\boldsymbol{A}_{\mathrm{ext}}=0$:

$$M^{(2)} \approx \mathrm{i}\Sigma_{fi}T^{\mathrm{pair}} \tag{3.28}$$

with

$$T^{\mathrm{pair}} = \langle R_f | -rU'/6m | R_i \rangle \tag{3.29}$$

and $U=-eA^0(\boldsymbol{x})$. The three terms (3.26), (3.27) and (3.29) just coincide with those in (2.22), obtained by starting with the Dirac amplitude (2.15). We now see that the rV' term in (2.22) has a simple physical interpretation—it represents corrections arising from the creation or destruction of free virtual pairs by the external potential.

3.2.4. Extension to He-like ions. Having laid the groundwork we can be fairly concise in sketching the theory of the $2^3S_1 \rightarrow 1^1S_0+\gamma$ decay in the He-like ions, emphasising only the new features which enter. The initial and final wavefunctions ψ_i and ψ_f now satisfy (3.17). The contributions to the amplitude M can be organised according to time-ordered diagrams which represent various terms in the expansion (3.20) and simple rules (Sucher 1958a; see Kelsey (1976) and Douglas and Kroll (1974)) can be

† We omit the details because the result can be regarded as a limiting case of the calculations carried out in §6, when $m_2 \rightarrow \infty$.

used to write down the integral associated with such a diagram. For the problem at hand the diagrams of immediate interest are shown in figures 3 and 4.

From figure 3(*a*) one has, in analogy with figure 2(*a*) and (3.22):

$$M_{3a}=2\langle\psi_f|\zeta_1|\psi_i\rangle \tag{3.30}$$

where we have introduced the abbreviation:

$$\zeta_1\equiv\boldsymbol{\alpha}_1.\boldsymbol{\epsilon}^*\exp(-i\boldsymbol{k}.\boldsymbol{r}_1).$$

From figure 3(*b*) one gets, neglecting electron binding energies relative to the energy $k'\sim\alpha m$ of the virtual photon:

$$M_{3b}\approx 2\langle\psi_f|\zeta_1 G_0(E_i)\,\Lambda_{++}B_{12}+B_{12}\Lambda_{++}G_0(E_f)\,\zeta_1|\psi_i\rangle \tag{3.31}$$

with $G_0(E)=(E-H_{++}+i\epsilon)^{-1}$ and B_{12} the Breit operator (3.4). The reduction to large components $\psi^{++}=(1+\beta_1)(1+\beta_2)\,\psi/4$ via:

$$\psi=(1+\boldsymbol{\alpha}_1.\boldsymbol{\xi}_1)(1+\boldsymbol{\alpha}_2.\boldsymbol{\xi}_2)\,\psi^{++}$$

can be carried out as in the hydrogenic case with the result that M_{3b} makes *no* contribution to M to order α^4, while M_{3a} takes the form, in analogy with figure 2(*b*) and (3.23):

$$M_{3a}\approx 2i\Sigma_{fi}T_{3a} \tag{3.32}$$

with

$$\Sigma_{fi}=\left\langle\chi_f\left|\,\boldsymbol{\sigma}_1.\frac{\boldsymbol{k}}{m}\times\boldsymbol{\epsilon}^*\,\right|\chi_i\right\rangle \tag{3.33}$$

and

$$T_{3a}=\langle\phi_f|\,\boldsymbol{p}_1^2/3m^2+k^2r_1^2/12\,|\phi_i\rangle. \tag{3.34}$$

Here χ is a two-electron spin wavefunction and $\phi=\phi(\boldsymbol{r}_1,\boldsymbol{r}_2)$ is an S-wave eigenfunction

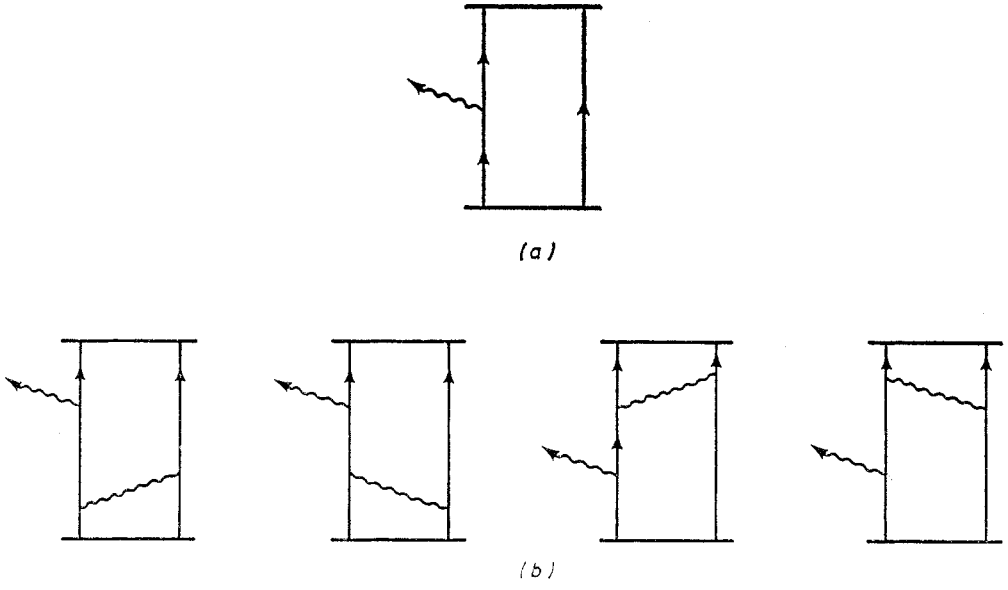

Figure 3. Time-ordered diagrams (*a*) and (*b*) corresponding to the amplitudes M_{3a} and M_{3b} of the text, respectively.

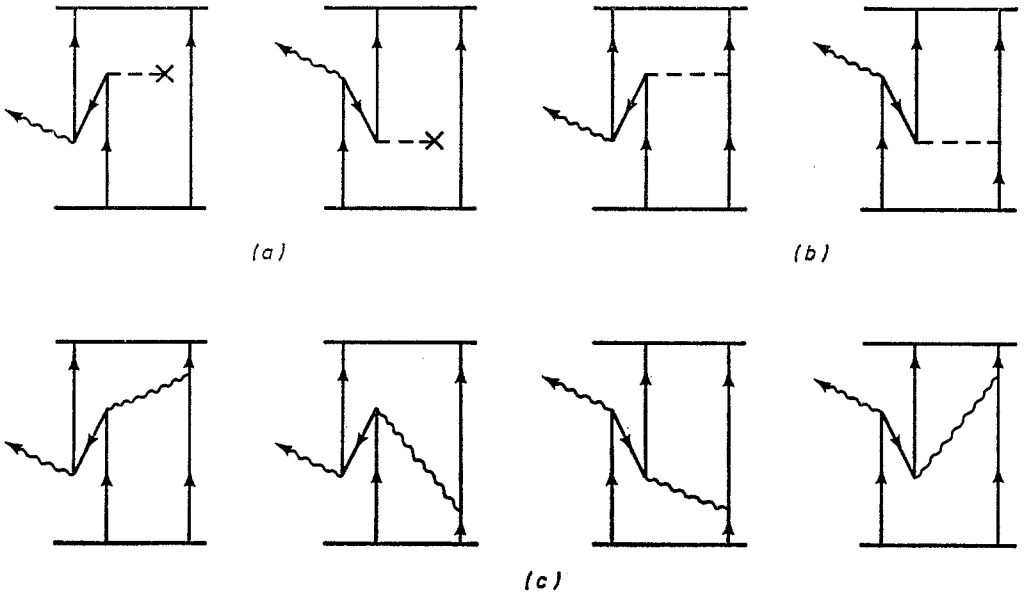

Figure 4. Time-ordered diagrams representing (*a*) one-pair, external Coulomb-field corrections, (*b*) one-pair, electron Coulomb-field corrections and (*c*) leading one-pair transverse-photon corrections, corresponding to the amplitude M_{4a}, M_{4b} and M_{4c} of the text, respectively.

of the completely non-relativistic Hamiltonian H_0:

$$H_0\phi = W\phi \qquad H_0 = \boldsymbol{p}_1^2/2m + \boldsymbol{p}_2^2/2m + U(r_1) + U(r_2) + V_{12}. \tag{3.35}$$

$\chi\phi$ is the non-relativistic limit of ψ^{++} and $k = W_i - W_f$.

Figures 4(*a*), (*b*) and (*c*) yield, on neglecting electron and photon energies relative to $2m$:

$$M_{4a} \approx 2\left\langle \psi_f \middle| \zeta_1 \Lambda_-(\boldsymbol{p}_1) \frac{U(r_1)}{2m} + \frac{U(r_1)}{2m} \Lambda_-(p_1)\, \zeta_1 \middle| \psi_i \right\rangle \tag{3.36}$$

$$M_{4b} \approx 2\left\langle \psi_f \middle| \zeta_1 \Lambda_-(\boldsymbol{p}_1) \frac{V_{12}}{2m} + \frac{V_{12}}{2m} \Lambda_-(\boldsymbol{p}_1)\, \zeta_1 \middle| \psi_i \right\rangle \tag{3.37}$$

$$M_{4c} \approx 2\left\langle \psi_f \middle| \zeta_1 \Lambda_-(\boldsymbol{p}_1) \frac{B_{12}}{2m} + \frac{B_{12}}{2m} \Lambda_-(\boldsymbol{p}_1)\, \zeta_1 \middle| \psi_i \right\rangle. \tag{3.38}$$

The reduction to the non-relativistic wavefunctions gives:

$$M_4 = M_{4a} + M_{4b} + M_{4c} \approx 2\mathrm{i}\Sigma_{fi}(T_{4a} + T_{4b} + T_{4c}) \tag{3.39}$$

where:

$$T_{4a} = -\langle \phi_f | r_1 U'(r_1)/6m | \phi_i \rangle \tag{3.40}$$

$$T_{4b} = -\langle \phi_f | \boldsymbol{r}_1 . \hat{r}_{12} V_{12}'(r_{12})/6m | \phi_i \rangle \tag{3.41}$$

$$T_{4c} = -\langle \phi_f | \boldsymbol{r}_1 . \hat{r}_{12} (\alpha r_{12}^{-2}/6m | \phi_i \rangle. \tag{3.42}$$

We have written T_{4b} in a form valid for an arbitrary static electron–electron interaction V_{12} in order to highlight the fact that with $V_{12} = \alpha/r_{12}$—the physical case—the

Coulomb pair term T_{4b} and the transverse photon pair term T_{4c} *cancel*, and only the external field pair term T_{4a} survives. Addition of (3.32) and (3.39) yields:

$$M \approx 2i\Sigma_{fi} T_{fi} \tag{3.43}$$

with

$$T_{fi} = \langle \phi_f | \, \boldsymbol{p}_1^2/3m^2 + k^2 r_1^2/12 - r_1 U'(r_1)/6m \, | \phi_i \rangle. \tag{3.44}$$

A short calculation then gives for the $2^3S_1 \rightarrow 1^1S_0 + \gamma$ decay rate:

$$R_{\text{He-like}} = \frac{2}{3} \frac{8\alpha k^3}{m^2} \, |T_{fi}|^2. \tag{3.45}$$

It is instructive to compare (3.45) with the hydrogenic rate (2.26) for large Z. If one uses product wavefunctions (suitably symmetrised) one finds $\sqrt{2}\, T_{fi} \rightarrow T_{\text{D}}$, defined by (2.22), and $k \rightarrow 3(\alpha Z)^2/8m$ so that

$$R_{\text{He-like}} = \tfrac{2}{3} R_{\text{H-like}}[1 + \mathcal{O}(Z^{-1})] \tag{3.46}$$

where $R_{\text{H-like}}$ denotes the hydrogenic rate (2.27). The factor $\frac{2}{3}$, which is the ratio of the statistical weight of the $^2S_{1/2}$ state in H to that of the $^3S_1$ state in He, may be simply understood as follows. In H-like ions the rate for a spin-flip transition ($s_z = \frac{1}{2} \rightarrow s_z = -\frac{1}{2}$) is *twice* that of the rate for a non-spin-flip transition ($s_z = \frac{1}{2} \rightarrow s_z = \frac{1}{2}$). In He-like ions only the spin-flip process is available, because of the Pauli exclusion principle—the outer electron with spin up can only join the spin-up electron in the inner shell by flipping its spin. Thus, if electron–electron interaction is neglected, the ratio of the rates is $2:(2+1) = \frac{2}{3}$.

The result (3.44)–(3.45) derived here from quantum electrodynamics (Feinberg and Sucher 1971) was obtained independently by Drake (1971), who used the Dirac–Breit Hamiltonian (3.1) and semiclassical radiation theory. Another treatment, also based on (3.1), was given by Beigman and Safranova (1971). (See Drake (1973) for a review and Drake (1972), Sucher (1976) and Lin (1977) for further discussion of the approach used by Drake (1971).)

3.2.5. A period of uncertainty. A systematic evaluation of the decay rate (3.45) for two-electron ions with large Z was first carried out by Drake (1971), who used many-parameter wavefunctions obtained by an expansion in powers of Z^{-1}. For $Z = 18$, corresponding to Ar^{16+}, he found $\tau = 212{\cdot}7$ ns to be compared with $\tau_{\text{exp}} = 172 \pm 30$ ns, as measured by Marrus and Schmieder (1970a,b). To quote from Drake's review of the subject at the *3rd International Atomic Physics Conference* (Drake 1973): 'The discrepancy, if real, is one of the few remaining examples of a disagreement in one- and two-electron systems . . .'.

As is well known, discrepancies are a Good Thing, providing a stimulus to further work by both experimentalists and theorists. This discrepancy was no different. On the experimental side, a number of new experiments on other species—S, Ti, Va and Fe—gave results which were largely consistent, within the errors, with the predictions of (3.45). However, a new discrepancy emerged for the case of Cl^{15+}, while the error bar for Ar^{16+} grew smaller. On the theoretical side, a number of new calculations were therefore undertaken.

(*a*) *Radiative corrections*—self-energy and vertex effects—were expected to be of the order of $\alpha \log (Z\alpha)^{-1}$, relative to the leading matrix element (3.43) (Feinberg and Sucher 1971). If the coefficient were positive and as large as say 2 or 3, then for

$Z \sim 20$ one would get a decrease of 10% or so in the lifetime. However, a detailed analysis for the case of H-like ions showed that although individual diagrams give $\alpha \log (Z\alpha)^{-1}$ corrections, these *cancel* on addition (Lin and Feinberg 1974, Lin 1975; see also Drake 1974). The correction from the anomalous electron magnetic moment, a factor of $(1+\alpha/2\pi)$, is also too small to matter—in fact, for the case of the $2^2S_{1/2} \rightarrow 1^1S_{1/2}+\gamma$ transition in H-like ions it is cancelled by another $\mathcal{O}(\alpha)$ effect so that there are no corrections of relative order α at all! In He-like ions there is a surviving radiative correction factor $(1+0{\cdot}97\alpha/\pi Z)$ which is, however, much too small to be of interest at present (Lin 1975). The formulae used by Lin and Feinberg (1974) as the starting point of their calculations were written down on heuristic grounds; their correctness was confirmed by a general analysis of radiative corrections to atomic decay rates which starts from first principles (Barbieri and Sucher 1977).

(*b*) *Recoil and retardation corrections* to the approximations made in arriving at (3.43), especially in diagrams involving transverse-photon exchange, were examined by Kelsey (1974, 1976). Although there are effects of relative order α arising from individual diagrams, these cancel on addition. Thus these corrections are, at best, of the relative order of $\alpha^2 Z$ and therefore again unimportant. Corrections arising from the hyperfine interaction are also negligible (Drake 1971).

(*c*) *Relativistic corrections* of the relative order of $(\alpha Z)^2$, which are present even if electron–electron interaction is neglected, need to be considered. For the hydrogenic $2^1S_{1/2} \rightarrow 1^1S_{1/2}+\gamma$ transition these may be found by keeping the next terms in an expansion in powers of αZ of the energy difference k and of the radial wavefunctions g and f, entering (2.17). The result of such a computation (Lin 1975) is that the RHS of (2.27) should be multiplied by a factor:

$$1+c_2(\alpha Z)^2 \tag{3.47}$$

where

$$c_2 = 1{\cdot}07. \tag{3.48}$$

Even for $Z=20$ this represents only a 2% correction to the rate and the corresponding correction for He-like ions will be equally small.

For the hydrogenic case, as mentioned earlier, M_D may in fact be evaluated exactly in terms of hypergeometric functions (Johnson 1972). Comparison of the exact values with those obtained via (2.27) and (3.47) shows that the terms of the order of $(\alpha Z)^4$ and higher are negligible even for Z as large as 40; this fact will be relevant for us shortly. Expansion of the exact formula for M_D yields (W R Johnson 1976 private communication):

$$c_2 = (19/20)+\ln(9/8) \tag{3.49}$$

which implies $c_2 = 1{\cdot}0678$, in agreement with (3.48).

Work to include such higher-order αZ effects for the He-like decays has been based on finding self-consistent relativistic Hartree–Fock wavefunctions, starting with the relativistic two-body Dirac–Coulomb Hamiltonian:

$$H_{DC} = \sum_{i=1}^{2} (\boldsymbol{\alpha}_i \cdot \boldsymbol{p}_i + \beta_i m - Z\alpha/r_i) + \alpha/r_{12} \tag{3.50}$$

and then directly evaluating the matrix element (3.22) with these wavefunctions. The results found this way (Johnson and Lin 1974; see also Feneuille and Koenig 1972) agree substantially with those found by the evaluation of the reduced form of the matrix element with the non-relativistic variational wavefunctions. From the above

remarks concerning the smallness of such effects in the H-like case, this is not surprising. (As already mentioned, results obtained from Hamiltonians such as (3.50) must in any case be treated with caution.)

(*d*) *The accuracy of the wavefunctions* used by Drake in his evaluation of (3.44) and (3.45) remained, at this stage, the only aspect of the calculation which had not been fully nailed down. Experience with another problem—the exact evaluation of the London–van der Waals constant C_{HH}, entering the H—H potential C_{HH}/r^6 (O'Carroll and Sucher 1968)—suggested that the first-order contribution of the electron–electron Coulomb interaction ought to be exactly calculable. The result of such a calculation is (Kelsey 1974, Kelsey and Sucher 1975, P J Mohr 1975 private communication):

$$R_{\text{He-like}} = R_{\text{He-like}}^{(0)}(1 + b_1/Z + b_2/Z^2 + \ldots) \tag{3.51}$$

where b_1 is exactly expressible in terms of (many) hypergeometric functions and:

$$R_{\text{He-like}}^{(0)} \equiv \frac{2}{3} R_{\text{H-like}}^{(0)} = \frac{2}{3} \frac{\alpha(\alpha Z)^{10}}{972} m. \tag{3.52}$$

Numerical evaluation of the lengthy formula for b_1 yields:

$$b_1 = -4{\cdot}099. \tag{3.53}$$

This is in extremely close agreement with the value of b_1 which can be inferred from Drake's variational calculation and confirms the accuracy of the wavefunctions used by him. The variationally determined value of b_2 is 6·7; we may use this to write:

$$R_{\text{He-like}} \approx R_{\text{He-like}}^{(0)}(1 - 4{\cdot}10/Z + 6{\cdot}7/Z^2). \tag{3.54}$$

The right-hand side of (3.54) reproduces the numerical values given by Drake (1971) to better than 1% for $Z > 16$; this is simply a consequence of the fact that the terms of order Z^{-3} or higher, as determined by the variational wavefunction, make only a very small contribution to $R_{\text{He-like}}$.

A completely different confirmation of the fact that the evaluation of equation (3.44) had been carried out to sufficient accuracy was obtained by Anderson and Weinhold (1975). Using theorems which enable one to put bounds on the error made in evaluating an off-diagonal element of an operator when approximate eigenfunctions are employed, these authors were able to show that, in the case of Ar^{16+}, the value obtained by Drake was good to better than 0·3%.

As the cloud on the QED horizon was being darkened by the apparent absence of large corrections to the rates computed from (3.44), it was discovered that remeasurement of the decay rate for Cl^{15+} considerably further downstream from the foil where the 2^3S_1 state is produced gave a different value for the lifetime, which moreover agreed with the theoretical value much better than that found previously; this had come from positions corresponding to only a fraction of the lifetime (Bednar *et al* 1975). A similar result was obtained on remeasurement of the Ar^{16+} lifetime (H Gould and R Marrus 1976 private communication). The reason for the seemingly non-exponential character of these decays has not been fully established. Quite recently it has been suggested (Armstrong and Lin 1977) that photons coming from an admixture of Li-like ions in the beam are responsible for this feature of the data. We shall give a detailed comparison between theoretical and experimental values in the next subsection, under the assumption that the earlier values of the Cl^{15+} and Ar^{16+} lifetimes are indeed artifacts of the experimental arrangement.

3.3. Comparison of theory with experiment

A systematic comparison of theory with experimental results from beam–foil spectroscopy may be carried out as follows. We observe first that the theoretical results described in the last subsection may be used to write an *explicit* formula for the M1 decay rate of the 2^3S_1 state which can be expected to be of high accuracy for $Z \gg 1$ and $(\alpha Z)^2 \ll 1$. To see this, let us write, for the hydrogenic decays:

$$R_{\text{H-like}}^{\text{th}} = R_{\text{H-like}}^{(0)}[1 + c_2(\alpha Z)^2 + d_1\alpha \log (Z\alpha)^{-1} + d_2\alpha + \Delta]. \tag{3.55}$$

The c term is the leading non-radiative correction to the lowest-order result (2.27), involving higher-order relativistic and retardation effects and relativistic corrections to the energy difference. The d terms represent the 'leading' radiative corrections. The remainder Δ consists of terms at most of the order of $(\alpha Z)^4$ or of the order of $\alpha(\alpha Z) \log (Z\alpha)^{-1}$ and so can be expected to be less than 1% even for, say, Z as large as 40. Let us further write, for the He-like decays:

$$R_{\text{He-like}}^{\text{th}} = R_{\text{H-like}}^{\text{th}}\, \eta \tag{3.56}$$

where η is a correction factor. To lowest order in α, the correction factor η can be read off from equation (3.51):

$$\eta = \tfrac{2}{3}(1 + b_1/Z + b_2/Z^2 + \ldots). \tag{3.57}$$

Using (2.27) and (3.47) we see that:

$$R_{\text{He-like}}^{\text{th}} = \tfrac{2}{3}R_{\text{H-like}}^{(0)}[1 + b_2/Z + b_2/Z^2 + c_2(\alpha Z)^2 + d_1\alpha \log (Z\alpha)^{-1} + d_2\alpha + \mathcal{O}(Z^{-3}) + \mathcal{O}(\alpha^2 Z) + \mathcal{O}(\alpha Z^{-1})]. \tag{3.58}$$

For the case at hand, we have, as discussed above, the *exact* results $d_2 = d_1 = 0$ (Lin and Feinberg 1974), $c_2 = 1{\cdot}07$ (Lin 1975, WR Johnson 1976 private communication) and $b_1 = -4{\cdot}10$ (Kelsey and Sucher 1975, PJ Mohr 1975 private communication). Further, $b_2 \approx 6{\cdot}7$, according to the variational calculation (Drake 1971). Thus, putting all this theoretical piecework together we can write the cooperative formula:

$$R_{\text{He-like}}^{\text{th}} \approx \frac{2}{3}\frac{\alpha(\alpha Z)^{10}\, m}{972}[1 - 4{\cdot}10/Z + 6{\cdot}7/Z^2 + 1{\cdot}07(\alpha Z)^2] \tag{3.59}$$

and the lifetime is:

$$\tau_{\text{He-like}}^{\text{th}} = (R_{\text{He-like}}^{\text{th}})^{-1}. \tag{3.60}$$

We expect, conservatively, that (3.59) is accurate to better than 2% in the entire range $15 \lesssim Z \lesssim 40$.

Table 1 shows the comparison between τ^{exp} and τ^{th} as computed from (3.59) and (3.60). As can be seen, the agreement between theory and experiment is very satisfactory. It should be noted that the experimental accuracy is not yet sufficient to provide a stringent test of the coefficients of the Z^{-2} and $(\alpha Z)^2$ terms. However, inclusion of these terms improves the agreement between the theoretical value and the central experimental value for *all* the accurately measured lifetimes. It will be quite interesting to see how (3.59) holds up—if and when the experimental accuracy reaches the 1% level. It seems clear that (3.59) may be used with confidence to compute $R(2^3S_1 \to 1^1S_0 + \gamma)$ for He-like ions in the indicated range. For Z larger than 45 or so, when the corrections of relative order $(\alpha Z)^4$ and higher begin to become

Table 1. Experimental and theoretical lifetimes for the decay $2^3S_1 \to 1^1S_0 + \gamma$ in He-like ions. For $Z = 16$ to 36, τ^{th} was computed from (3.60) of the text; for $Z = 2$, τ^{th} is taken from (3.62) of the text.

Element	Z	τ^{exp} (ns)	τ^{th} (ns)	τ^{exp}/τ^{th}	Reference for τ^{exp}
S	16	706 ± 86	697	$1{\cdot}01 \pm 0{\cdot}12$	Bednar *et al* (1975)
Cl	17	354 ± 24	373	$0{\cdot}95 \pm 0{\cdot}06$	Bednar *et al* (1975)
A	18	202 ± 20	207	$0{\cdot}98 \pm 0{\cdot}10$	Gould and Marrus (1976 private communication)
Ti	22	$25{\cdot}8 \pm 1{\cdot}3$	26·4	$0{\cdot}98 \pm 0{\cdot}05$	Gould *et al* (1973)
V	23	$16{\cdot}9 \pm 0{\cdot}7$	16·8	$1{\cdot}01 \pm 0{\cdot}04$	Gould *et al* (1974)
Fe	26	$4{\cdot}8 \pm 0{\cdot}6$	4·78	$1{\cdot}00 \pm 0{\cdot}13$	Gould *et al* (1974)
Kr	36	$0{\cdot}20 \pm 0{\cdot}06$	0·170	$1{\cdot}18 \pm 0{\cdot}35$	Gould and Marrus (1976)
He	2	$0{\cdot}91^{+0{\cdot}40}_{-0{\cdot}20} \times 10^4$ s	$0{\cdot}841 \times 10^4$ s	$1{\cdot}08^{+0{\cdot}48}_{-0{\cdot}24}$	Woodworth and Moos (1975)

significant, one could use the approximation:

$$R_{\text{He-like}} \approx \tfrac{2}{3} R_{\text{H-like}}^{(0)} (1 - 4{\cdot}1\, Z^{-1})$$

where $R_{\text{H-like}}^{(0)}$ denotes the rate for $2^1S_{1/2} \to 1^1S_{1/2} + \gamma$ in H-like ions with radiative corrections neglected, which has been evaluated exactly for an external Coulomb field (Johnson 1972). For $Z < 15$ the $(\alpha Z)^2$ correction is less than 1% and here the rates computed by Drake (1971) should be sufficiently accurate, at least for Z not so small that a Z^{-1} expansion for the wavefunctions is only slowly convergent. Of course, for $Z = 2$ very accurate Hylleraas-type wavefunctions can be used. By way of summary, the significant decay modes of the 2^1S_0 and 2^3S_1 states of He-like ions are shown in figure 5.

As mentioned earlier, there is now available a much improved measurement of the rate for the $2^3S_1 \gg 1^1S_0 + \gamma$ transition in helium itself, viz $(\tau^{exp})^{-1} = 1{\cdot}10 \times 10^{-4}\ s^{-1} \pm 30\%$ (Woodworth and Moos 1975) giving the value quoted in table 1:

$$\tau^{exp} = (0{\cdot}91^{+0{\cdot}40}_{-0{\cdot}20}) \times 10^4 \text{ s}. \tag{3.61}$$

This agrees quite well with

$$\tau^{th} = 0{\cdot}841 \times 10^4 \text{ s} \tag{3.62}$$

obtained from a six-parameter Hylleraas-type wavefunction (Feinberg and Sucher 1971).

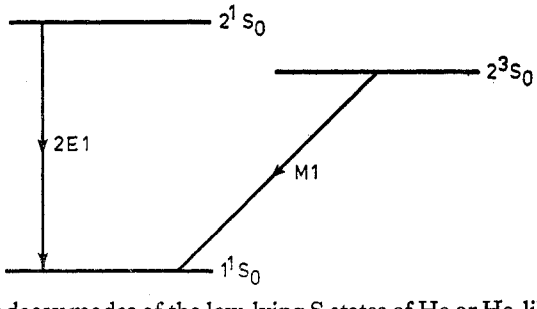

Figure 5. Dominant decay modes of the low-lying S states of He or He-like ions: $2^1S_0 \to 1^1S_0 + 2\gamma$ and $2^3S_0 \to 1^1S_0 + \gamma$.

Thus, the theoretical and experimental values are in agreement over a range covering fourteen orders of magnitude. This is surely an impressive triumph for QED in a domain rather different from that in which it has been tested heretofore.

We conclude this discussion of relativistic magnetic dipole transitions in one- and two-electron atoms with two comments. First, it seems ironic that the clearest evidence for these transitions and confirmation of the theory comes from the He-like ions rather than the simpler H-like ions. The reason is that

$$R_{\text{H-like}}(2^1S_{1/2}\rightarrow 1^1S_{1/2}+\gamma)=2{\cdot}50\,Z^{10}\times 10^{-6}\ \text{s}^{-1}$$

is only a few per cent of $R_{\text{H-like}}(2^1S_{1/2}\rightarrow 1^1S_{1/2}+2\gamma)\approx 8\,Z^6\ \text{s}^{-1}$ even for Z as large as twenty and does not dominate until $Z>45$.

Second, from a theoretical point of view, it may perhaps be regarded as disappointing that the electron–electron interaction, especially its short-distance behaviour, does not play a more significant role in determining the rate. However, it is gratifying that to leading order the rate for the M1 decay in the more complicated He-like ions is nevertheless expressible in closed form, modulo the non-relativistic wavefunctions.

We turn now to a recently discovered set of particles which appear to be describable as bound states of more elementary constituents and which, like the states of an atom, are connected by radiative transitions. To the extent that a non-relativistic description of the bound states is valid, some of these transitions may be characterised as 'relativistic M1 transitions' and these can be studied by a straightforward generalisation of the techniques described above. In contrast to the atomic case, the corresponding amplitudes are very sensitive to the details of the interaction between the constituents. Of course, quite apart from this, the system to be discussed has charms of its own.

4. Narrow resonances at 3 GeV: discovery and theoretical background

4.1. The discovery of the psions

Let us review the developments which about three years ago electrified the community of elementary-particle physicists. In November 1974 the following discoveries were announced.

(i) A high and narrow peak in the e^+e^- invariant mass spectrum was observed by an MIT–BNL group (Aubert *et al* 1974) in the reaction $p+Be\rightarrow e^+e^-+X$. The peak mass was at 3·1 GeV and the width was less than the energy resolution of 20 MeV, 'consistent with zero'.

(ii) Extremely high and sharp peaks in the elastic and various inelastic e^+e^- cross sections were observed by a SLAC group (Augustin *et al* 1974) in colliding-beam experiments at the SPEAR facility. These peaks were seen in the Bhabha scattering $e^+e^-\rightarrow e^+e^-$, in $e^+e^-\rightarrow\mu^+\mu^-$, and in $e^+e^-\rightarrow$hadrons, at a CM energy of $3{\cdot}105\pm 0{\cdot}003$ GeV, with an upper limit of 1·3 MeV for the full width at half-maximum.

These discoveries were confirmed within a week by three groups working at Frascati (Bacii *et al* 1974).

The natural interpretation of the observation at BNL was that it corresponded to the production of a new particle or resonance—called J—followed by its decay into e^+e^-:

$$p+Be\rightarrow J+\text{anything}\qquad J\rightarrow e^+e^-. \tag{4.1}$$

Similarly, the SLAC experiments corresponded to the formation of a new particle—called ψ—followed by its decay into both leptonic and hadronic channels:

$$e^+e^- \to \psi \to e^+e^-,\ \mu^+\mu^-,\ \text{hadrons}. \tag{4.2}$$

Of course, because of the coincidence of the mass and the common e^+e^- decay channel, J and ψ were immediately regarded as the same particle, now often designated as J/ψ.

(iii) A week after the discovery of the ψ, the SLAC group found another narrow resonance similar to the ψ, called ψ', at 3684 MeV (Abrams *et al* 1974). Analysis of the final states showed an important decay channel for the ψ' to be

$$\psi' \to \psi + \text{hadrons}. \tag{4.3}$$

There followed a period of intense experimental activity accompanied by equally intense theoretical analysis and speculation. The excitement of this period can be recaptured by perusal of the proceedings of the *1975 International Symposium on Lepton and Photon Interactions at High Energies* (Kirk 1976), held at SLAC. The remarkable aspect of the ψ was that, unlike other states coupled to both leptons and hadrons, such as the ρ, ω and ϕ mesons, which have *total* widths in the 5–150 MeV range, but purely leptonic widths of the order of 1–10 keV, the ψ had a total width of the order of only 0·1 MeV, although its leptonic width was 'normal'.

Analysis of the experiments showed that the ψ and ψ' could be regarded as *hadrons*, with the same quantum numbers as the photon, $J^{PC} = 1^{--}$, i.e. with angular momentum $J=1$, odd under space inversion P and odd under charge conjugation C, as one might naively guess. This assignment was confirmed by observation of interference just below resonance of the amplitudes $A(e^+e^- \to \gamma \to \mu^+\mu^-)$ and $A(e^+e^- \to \psi \to \mu^+\mu^-)$. Study of the hadronic final states showed consistency with an assignment $I=0$ and $G=-1$ for the isospin and G parity of both the ψ and ψ'.

4.2. The three c's: charm, colour and confinement

4.2.1. Charm. The first detailed dynamical models for the psions, ψ and ψ', involved the notion of *charm*. What is charm? A good characterisation was actually given long ago: '. . . it's a sort of bloom . . . if you have it, you don't need to have anything else; and if you don't have it, it doesn't much matter what else you have' (Barrie 1918).

A more technical definition of charm is that it is a quantum number, denoted by 'C', which is assigned to the strongly interacting particles, together with the familiar quantum numbers such as the electric charge quantum number Q, the baryon number B, and the strangeness S or equivalently the hypercharge $Y=B+S$. Like Y, C is imagined to be conserved by strong as well as electromagnetic interactions but not by weak interactions. The introduction of such a new quantum number or 'flavour', as the current jargon has it, was originally motivated by relatively slender, more or less aesthetic considerations (Bjorken and Glashow 1964, Hara 1964; see also Tarjanne and Teplitz 1963). There were two routes to charm, one by way of the strong interactions or the 'front door', the other by way of the weak interactions, or 'back door', in terminology once used by B d'Espagnat in discussing approaches to the symmetries of strong interactions.

To describe these routes, recall that in 1961 it was discovered that the invariance of strong interactions under SU(2), with generators I_i ($i=1, 2, 3$), which allowed the then known hadrons to be organised into isospin multiplets, could be enlarged to an approximate invariance under the group SU(3), with generators F_i ($i=1, \ldots, 8$) which allowed organisation of the hadrons into SU(3) 'supermultiplets'. The states within a supermultiplet could be labelled by the eigenvalues of two commuting generators F_3 and F_8 or equivalently by I_3 and Y, on making the identification $I_3=F_3$, $Y=2F_8/\sqrt{3}$. Although the known hadrons could be successfully fitted into eight- and ten-dimensional SU(3) multiplets, **8** and **10**, only *three* hadrons were needed to carry the conserved quantum numbers B, Q and S. The most economical scheme was that introduced independently by Gell-Mann (1964) and Zweig (1964), who imagined the existence of a fundamental triplet of spin-$\frac{1}{2}$ fermions q_1, q_2, q_3 called 'quarks', each with $B=\frac{1}{3}$, but with (q_1, q_2) an isospin doublet (u, d) of 'up' and 'down' quarks with $S=0$ ($Y=\frac{1}{3}$) and q_3 an isosinglet 'strange' quark 's', with $S=-1$ ($Y=\frac{1}{3}$). All known mesons and baryons could then be considered, at least on a formal level, as composites $q\bar{q}$ and qqq of the q_i and their antiparticles $\bar{q}_i$, which carry the fundamental representations **3** and **3*** of SU(3).

Recall also that by 1963 the known low-energy weak interactions could be successfully described by an effective interaction of the current–current type, $\mathscr{H}_W^{\text{eff}}=(G_F/\sqrt{2})\, J^\alpha J_\alpha{}^\dagger$ where $J_\alpha=l_\alpha+h_\alpha$ was the sum of a leptonic current $l_\alpha(x)=\bar{e}(x)\,\Gamma_\alpha\nu_e(x)+\bar{\mu}(x)\,\Gamma_\alpha\nu_\mu(x)$ (with, for example, $\nu_e(x)$ denoting the Dirac field associated with the electron neutrino ν_e and $\Gamma_\alpha=\gamma_\alpha(1-\gamma_5)$) and a hadronic current $h_\alpha(x)$. As far as its transformation properties under SU(3) were concerned, $h_\alpha(x)$ could be written in terms of hypothetical quark fields $u(x)$, $d(x)$ and $s(x)$ as $h_\alpha(x)=\bar{d}_c(x)\,\Gamma_\alpha u(x)$ where $d_c=d\cos\theta_c+s\sin\theta_c$ was one of two orthogonal fields obtained by a rotation with angle θ_c (the Cabibbo angle) from $d(x)$ and $s(x)$, the other being $s_c=s\cos\theta_c-d\sin\theta_c$ (Cabibbo 1963).

Front door. With this as background, we can describe the strong interaction route to charm as coming from the observation that the mass spectrum of the vector meson $J^{PC}=1^{--}$ resonances was suggestive of a (badly broken) symmetry of the strong interactions which was higher than SU(3) and SU(4) was a natural candidate. But SU(4) has *three* commuting generators, two of which could be identified with $F_3=I_3$ and $F_8\propto Y$, leaving a third to be identified with some as yet (at the time) undiscovered quantum number 'C' which was zero for the known particles.

Back door. The weak interaction route to charm came from considerations of *lepton–hadron symmetry*, within the framework of the quark model. For many years it was assumed, basically for simplicity, that the electron neutrino and muon neutrino were the same: $\nu_e=\nu_\mu=\nu$. There seemed then to be a nice parallel between the three leptons (ν, e, μ) and the three quarks (u, d, s) (or (P, N, Λ) in the Sakata model) and the pieces $(\bar{e}\Gamma_\alpha\nu)$, $(\bar{\mu}\Gamma_\alpha\nu)$ entering the leptonic current l_α seemed analogous to the pieces $(\bar{d}\Gamma_\alpha u)$, $(\bar{s}\Gamma_\alpha u)$ entering the hadronic current h_α. However, when experiment showed that $\nu_e\neq\nu_\mu$ (Danby *et al* 1962) this analogy was lost. The introduction of a fourth quark c, consistent with an enlargement from SU(3) to SU(4), allowed restoration of equality between numbers of leptons and quarks. Moreover if, for example, c were assigned the same charge as u, the similarity between l_α and h_α could be restored by adding a piece $(\bar{s}_c\Gamma_\alpha c)$ to h_α, with the pleasing correspondence $(\bar{e}\Gamma_\alpha\nu_e)\leftrightarrow(\bar{d}_c\Gamma_\alpha u)$ and $(\bar{\mu}\Gamma_\alpha\nu_\mu)\leftrightarrow(\bar{s}_c\Gamma_\alpha c)$. With the assignment $S=0$ and $C=1$ for 'c', the Gell-Mann–Nishijima formula $Q=I_3+Y/2$ then generalises to $Q=I_3+Y/2+C/2$, and by analogy with the SU(3) quark model one would expect to eventually

observe composite systems such as $D^+ = \bar{d}c$ which would be a 'charmed meson' with $S=0$ but $C=1$, or systems such as $\bar{c}c$, with $C=0$ but with 'hidden charm'.

The motivation for a fourth quark was considerably strengthened by the observation that it could play a role in avoiding violation by higher-order (divergent) corrections of the selection rules built into $\mathscr{H}_W^{\text{eff}}$ (Glashow *et al* 1970). It received a major boost after the discovery of renormalisable, unified gauge theories of weak and electromagnetic interactions, based on the Yang–Mills idea of local gauge invariance of a field theory under a non-Abelian group and the Higgs–Kibble mechanism for the spontaneous symmetry breaking of the local symmetry. This came from the observation that the introduction of such a quark allows one to accommodate in an elegant way, consistent with the ideas of unified gauge theories, both (i) the suppression of strangeness-changing neutral currents, inferred from the smallness of the $K_1^0 - K_2^0$ mass difference and the low rate of, for example, $K_L^0 \to \mu^+\mu^-$, and (ii) the presence of neutral currents as eventually observed in strangeness-non-changing high-energy neutrino interactions. (For reviews see Abers and Lee (1973), Bég and Sirlin (1974), Bernstein (1974) and Weinberg (1974).)

4.2.2. Colour and confinement. Fractionally charged objects such as the quarks of Gell-Mann and Zweig, at least one of which must be stable if the conservation of charge is absolute, have been the object of experimental searches for about fifteen years, without success. (The most recent review of how and where quarks have *not* been found is Jones (1977); the evidence for fractional charge obtained recently by LaRue *et al* (1977) remains controversial (see, for example, Morpurgo 1977, Fairbank 1977)). Another thread in the story must now be drawn in. Exploration in the mid-sixties of the mass spectrum of the hadrons revealed the existence of an approximate SU(6) invariance of the strong quark-binding interactions, corresponding to joint transformations of the SU(3) indices and the spin indices associated with the non-relativistic SU(2) spin group. This could be made compatible with Fermi statistics for the quarks if there were another, hidden, degree of freedom (Greenberg 1964), nowadays described as the introduction of a three-valued 'colour' index for the quark field, which can be used to antisymmetrise the baryon wavefunctions. In the so-called standard version of the gauge theories of weak, electromagnetic and strong interactions, the Lagrangian is imagined to be exactly invariant not only under global SU(3) transformations of the colour index, but also under *local* $SU(3)_{\text{colour}}$ transformations.

This invariance requires the existence of a set of eight massless vector mesons, called 'colour gluons', and one arrives at a theory now known as quantum chromodynamics (QCD), a non-Abelian gauge theory analogue of QED, which is an Abelian gauge theory. The theory has, on the one hand, a remarkable property called asymptotic freedom, which means that at short distances (high momentum transfers) the interaction between quarks arising from the exchange of colour gluons becomes weak and perturbation theory is applicable. At large distances, on the other hand, it is speculated that the coupling arising from colour gluon exchange *grows* with separation between quarks and that this leads not only to binding but to binding with unusual permanence. Because of the infrared complexities of such theories one is allowed to entertain the hope that the associated long-range forces will be such that only objects which are colour-neutral, i.e. singlets under $SU(3)_{\text{colour}}$, are observable as real particles, having asymptotic states associated with them in Hilbert space. Thus in this view both the colour-triplet quarks and the colour-octet gluons are forever out

of view, e.g. protected from leaving tracks in a bubble chamber. The masses of the quarks then become 'effective masses' or parameters which can be determined only indirectly by experiment.

4.3. Width of $(\bar{c}, c)$ bound states and the OZI rule

Although the ψ and ψ' could not themselves be charmed particles (more generally, could not possess a non-zero value for an additive quantum number conserved by strong and electromagnetic interactions) a number of theorists advocated the viewpoint that they are composites of a c and a $\bar{c}$ in the same sense that the 'classical' vector mesons are regarded as composites in the SU(3) quark model: $\phi = \bar{s}s$, $\omega = (\bar{u}u + \bar{d}d)/\sqrt{2}$, etc. In fact it was anticipated, in analogy with the ϕ meson, that narrow states symbolised by $\bar{c}c$ should exist (Appelquist and Politzer 1975a). This picture has an immediate qualitative appeal. It relates the narrow hadronic widths of the psions to the success of the Okubo–Zweig–Iizuka (OZI) rule or 'quark line rule' which states that in the SU(3) quark model decay processes are strongly suppressed if they can only proceed by the annihilation of a $q\bar{q}$ pair present in an initial-state hadron or by the creation of a pair which forms part (or all) of a final-state hadron. This empirical rule accounts, for example, for the fact that the ϕ, which is largely $s\bar{s}$, decays predominantly into $K\bar{K}$ rather than $\pi^+\pi^-\pi^0$, for which the phase space is much bigger; note that since the pions, unlike the K, contain *no* s or $\bar{s}$ quarks one can only get rid of the initial $\bar{s}s$ pair by mutual annihilation. An extended OZI rule would then inhibit the transformation of psions into ordinary hadrons, which have no c quarks in them. Provided that $c\bar{q}$ systems ('charmed' mesons) have masses above $\frac{1}{2}m_{\psi'} \sim 1{\cdot}85$ GeV there would be no open decay channels which would be allowed by the OZI rule. Of course, this does not by itself explain the narrow widths, but it does at least provide, one might say, a unification of two mysterious facts. On a more technical level, it was suggested that the property of asymptotic freedom in quantum chromodynamics might in fact account for the validity of an extended OZI rule when heavy quarks ($m_c \sim$ few GeV) are involved.

The idea that $m_c \sim$ few GeV arose in part from the study of strangeness-changing second-order weak processes within the framework of unified gauge theories. For example, if on the one hand, the c were degenerate in mass with the ordinary quarks the suppression of such effects would be too strong. On the other hand, an estimate of the $K_L K_S$ mass difference, which is proportional to $G_F \alpha m_c^2$ in the standard model, shows that m_c cannot be too large. In addition, one would expect to have already seen charmed hadrons unless m_c exceeded m_q by at least 1 GeV or so. Thus, before the discovery of the psions one had arrived at the conclusion that, roughly, $1{\cdot}5 \text{ GeV} \lesssim m_c \lesssim 5$ GeV (see Gaillard *et al* (1975) for a detailed review).

In these few pages we have of course been unable to do more than lightly scan the rich tapestry of ideas and concepts woven by theorists over a period of many years to describe and correlate the phenomena involved. Apart from those already mentioned, other reviews which the reader may wish to consult include Politzer (1974) (asymptotic freedom), Okubo (1978) (quark line rule), Greenberg and Nelson (1977) (colour models of the hadrons) and Marciano and Pagels (1978) (quantum chromodynamics). For a review of the early e^+e^- experiments see Feldman and Perl (1975).

Although we cannot explore these issues further here, we want to note, for use in the following subsection, that the idea that the quarks are permanently confined inside hadrons expresses itself in the modern quark-binding industry by way of the

assumption that the force between, say a q and a $\bar{q}$, *grows* with their separation r in such a way that it would require infinite energy to break them up. Thus, for large r, one assumes, typically, that $V_{q\bar{q}} \sim ar$ with $a > 0$. However, it should be emphasised that exact confinement is not essential for the potential model description of the psions to which we now turn. In particular, there exists a comprehensive unified gauge theory of weak, electromagnetic and strong interactions, compatible with present experimental knowledge, in which the basic hadronic building blocks have integer charge and are not confined but may instead have escaped detection so far because they are highly unstable. (For reviews see Pati and Salam (1977) and Pati (1976).) Furthermore, it is possible to develop an algebraic approach to the properties of the narrow resonances, based on commutators of vector and axial charges within the framework of 'asymptotic SU(4) symmetry', which does not make such explicit use of the quark concept (for a review see Oneda 1978).

5. The charmonium model

5.1. Description of the model

The bound states of a particle–antiparticle system such as $(c, \bar{c})$ can be classified by J^{PC} where J is the total angular momentum, and P and C are the eigenvalues of parity and charge conjugation operators, respectively. For a state which in non-relativistic spectroscopic notation is described by $^{2S+1}L_J$, P and C are related to the orbital and spin angular momentum quantum numbers L and S by:

$$P = -(-1)^L \qquad C = (-1)^{L+S}. \tag{5.1}$$

It follows from (5.1) that if the $(c, \bar{c})$ system has any triplet-S ($^3S_1$) bound states, they will have:

$$J^{PC} = 1^{--} \tag{5.2}$$

that is, precisely the quantum numbers of the ψ and ψ'. The basic hypothesis of most models of the psions as bound states is therefore:

(A) ψ is the ground state of 'orthocharmonium', i.e. can be identified with the lowest lying $^3S_1$ state of a bound $(c, \bar{c})$ system; ψ' is the first excited $^3S_1$ state, a radial excitation of ψ.

Tests of (A) alone would be difficult to come by. But the addition of further hypotheses leads to a number of predictions which can be tested. These hypotheses all concern the dynamics of the binding.

(B_1) The energy levels of the bound $(c,\bar{c})$ system may, to a first approximation, be found by solving a non-relativistic Schrödinger equation:

$$\left(\frac{\boldsymbol{p}^2}{2\mu} + V_b\right)\phi = W\phi \tag{5.3}$$

where $\mu = \frac{1}{2}m_c$ is the reduced mass, W is the 'binding energy', and $M = 2m_c + W$ is the total mass of the bound state with wavefunction ϕ.

(B_2) If the binding interaction V_b is written as the sum of a spin-independent part V_{si} and a spin-dependent part V_{spin}:

$$V_b = V_{si} + V_{spin} \tag{5.4}$$

then

$$V_{si} \gg V_{spin}. \tag{5.5}$$

(B_3) V_{si} is a local central potential, popularly taken to be of the form:

$$V_{si} = ar + b - (4/3)\,\alpha_s r^{-1} \tag{5.6}$$

with a and b parameters to be determined and α_s small, as will be explained later.

5.2. Qualitative predictions and comparison with experiment

Before discussing (B_3), we should emphasise that (B_1) and (B_2) alone already lead to certain qualitative predictions, regardless of the precise form of V_{si} (Appelquist *et al* 1975, Callan *et al* 1975).

One expects to find in the neighbourhood of ψ and ψ' other ψ-like states as follows.

(i) ψ and ψ' should have nearby singlet-S state ($^1S_0$) 'hyperfine' partners, the ground and first excited states of 'paracharmonium'. The quantum numbers of these states, denoted as η_c and η_c' respectively, would be 0^{-+}, according to equation (5.1).

(ii) There should be bound orbital excitations of at least the ψ, the lowest lying being P states. In particular, one expects to see a multiplet of triplet-P states: $^3P_0$, $^3P_1$ and $^3P_2$, with quantum numbers 0^{++}, 1^{++} and 2^{++}, respectively. We shall denote them by χ_J ($J = 0, 1, 2$).

These expectations are summarised in figure 6. The η_c and η_c' states have been drawn relatively near to and below the ψ and ψ' states corresponding not only to the assumption (B_2) that the spin-dependent forces are small, but also that they are attractive in singlet states, as in positronium. Since $C = +1$ for the η_c, the η_c' and the χ_J states, these will not show up as narrow resonances in e^+e^- annihilation, assumed to proceed via a one-photon intermediate state ($C = -1$). However, if these states exist, then *new radiative* decay modes become available to the ψ and ψ'. In

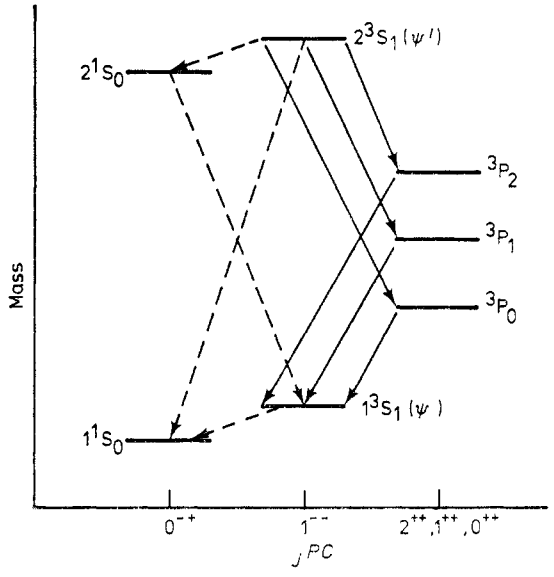

Figure 6. Schematic partial spectrum of charmonium and radiative decays expected on the basis of hypotheses (B_1) and (B_2) of the text. Full and broken arrows indicate E1 and M1 decays, respectively. Possible low-lying states $^1P_1$, $^1D_2$ and $^3D_1$ are omitted.

particular, if $m_\chi < m_{\psi'}$, as is the case if (B_3) holds, one may look for the E1 decays of the ψ':

$$\psi' \to \chi_J + \gamma. \tag{5.7}$$

Even if the photon is not directly observed, χ_J will itself have direct hadronic decays so that there should be a signal in those final hadronic channels that have $C = +1$. Furthermore, χ_J can itself undergo a radiative E1 transition to the ψ if $m_{\chi_J} > m_\psi$:

$$\chi_J \to \psi + \gamma. \tag{5.8}$$

So observation of two-photon events in the region of the ψ' mass might reveal the existence of the cascade process, (5.7) followed by (5.8).

In addition, both the ψ' and the ψ should undergo ordinary radiative M1 transitions to their $^1S_0$ partners:

$$\psi' \to \eta_c' + \gamma \tag{5.9}$$

$$\psi \to \eta_c + \gamma \tag{5.10}$$

and, finally, one ought to be able to observe the relativistic M1 transitions:

$$\psi' \to \eta_c + \gamma \tag{5.11}$$

$$\eta_c' \to \psi + \gamma. \tag{5.12}$$

So at last we return to the subject discussed at length in the atomic case. Although the designation of the decays (5.9)–(5.12) as M1 transitions is essentially model-independent, relying only on the fact that we have $\Delta J = 1$, and *no* change of parity, it should be emphasised that the use of the adjectives 'ordinary' and 'relativistic' is appropriate only if the non-relativistic description, in which the ψ and ψ' radial wavefunctions are orthogonal and η_c and η_c' have the same radial wavefunction as ψ and ψ' respectively, is indeed a good first approximation.

How have these qualitative predictions fared in the light of the intense experimental activity referred to above? After some initial disappointment, existence of the states of the predicted type, together with the existence of most of the predicted radiative decays, has largely been confirmed. From study of the final states in ψ' decay, convincing evidence has been found for the existence of three $C = +1$ states, denoted by $\chi(3{\cdot}41)$, $\chi(3{\cdot}51)$ and $\chi(3{\cdot}55)$, where the number in parentheses denotes the mass in GeV. In each case one has observed both the process:

$$\psi' \to \chi + \gamma \qquad \chi \to \text{hadrons} \tag{5.13}$$

and the two-photon cascade process:

$$\psi' \to \chi + \gamma_1 \qquad \chi \to \psi + \gamma_2 \tag{5.14}$$

with $\psi \to e^+e^-$. There is also (weaker) evidence for a fourth $C = +1$ state between the ψ and ψ', denoted by $\chi(3.45)$, seen in a process (5.14)—but not in (5.13). In addition, there is evidence from observation of three-photon final states in the decay of the ψ for a fifth $C = +1$ state, denoted by $X(2{\cdot}83)$, via:

$$\psi \to X + \gamma_1 \qquad X \to \gamma_2\gamma_3. \tag{5.15}$$

Like the $\chi(3{\cdot}45)$ the X has also not yet been seen in hadronic final states. (For a comprehensive review of the experiments and references, see Feldman and Perl (1977).)

Thus it seems very likely that at least five narrow C-even states have been identified in the range of interest. If one assumes that the four observed states between ψ and ψ' correspond in some order to the three $\chi_J = {}^3P_J$ and to $\eta_c' = 2{}^1S_0$ the most likely assignment compatible with all the data is (Chanowitz and Gilman 1976):

$$\chi(3{\cdot}41) \leftrightarrow \chi_0 \qquad \chi(3{\cdot}51) \leftrightarrow \chi_1 \qquad \chi(3{\cdot}55) \leftrightarrow \chi_2 \tag{5.16}$$

and

$$\chi(3{\cdot}45) \leftrightarrow \eta_c'. \tag{5.17}$$

The $X(2{\cdot}83)$ is the one and only candidate for the ground state of 'paracharmonium':

$$X(2{\cdot}83) \leftrightarrow \eta_c. \tag{5.18}$$

Quite recently the existence of the χ_J states has been confirmed by study of the inclusive γ-ray spectrum of the ψ' (Biddick *et al* 1977):

$$\psi' \rightarrow \gamma + \text{anything}. \tag{5.19}$$

Clear peaks are seen at $E_\gamma = 121$ MeV, 169 MeV and 260 MeV which are interpretable respectively as $\psi' \rightarrow \chi_J + \gamma$ ($J = 2, 1, 0$) as well as at $E'_\gamma = 450$, corresponding to the sequence $\psi' \rightarrow \chi_2 + \gamma$, $\chi_2 \rightarrow \psi + \gamma'$, with χ masses in good agreement with those given above.

Very recently a new psionic resonant state $\psi(3{\cdot}77)$ with a width of about 26 MeV has been reported (Rapidis *et al* 1977), which coincides very closely in mass with a ${}^3D_1$ state predicted from a potential model (Eichten *et al* 1976). If this state is indeed a ${}^3D_1$ state then the qualitative success of the idea that the resonant states observed in the 3–4 GeV region should be thought of as bound states of a spin-$\frac{1}{2}$ fermion and antifermion is almost complete, in the following sense. With $J \leqslant 2$, the J^{PC} states *not* allowed by (5.1) are 0^{--}, 0^{+-}, 1^{-+} and 2^{+-}, and there is no evidence for such states. The allowed states include 0^{++}, 0^{-+}, 1^{--}, 1^{++}, 2^{++} and 2^{--} for which there is evidence ranging from weak to definitive. Allowed states such as $1^{+-} = {}^1P_1$ and $2^{-+} = {}^1D_2$ are expected to be difficult to observe, even if they exist, because of selection rules. The situation is summarised in figure 7 adapted from Trippe *et al* (1977) who give a complete survey of experimental results to March 1977.

5.3. Quantitative aspects of the extreme non-relativistic model

The source of the potential (5.6) is as follows. Recall that in QED the exchange of a single photon between an electron and a positron, coming from the coupling $e\bar{\psi}\gamma^\mu\psi A_\mu$, gives rise to an interaction potential which in the static limit is just the Coulomb potential $-(e^2/4\pi)\, r^{-1}$. In QCD there is a similar coupling term

$$g_s \bar{q}\gamma^\mu (\lambda_a/2)\, q V_\mu{}^a$$

where λ_μ is an SU(3) matrix, $V_\mu{}^a$ is a massless colour gluon field and 'a' is summed from 1 to 8. Hence the static potential arising from colour gluon exchange between a c and a $\bar{c}$ is just $-(\lambda_a(1)\lambda_a{}^*(2)/4)(g_s{}^2/4\pi)\, r^{-1}$ and a little algebra shows that in a colour singlet state the value of the factor involving the λ is just 4/3. Thus, with α_s defined as $g_s{}^2/4\pi$, in analogy to $\alpha = e^2/4\pi$, we arrive at the last term in (5.6), which is intended to approximate the $c\bar{c}$ interaction at *short* distances where, because of asymptotic freedom, single-gluon exchange should be a good approximation. The first term in (5.6) is supposed to represent the confinement arising, presumably, from multi-gluon exchange which at *large* distances cannot be treated perturbatively.

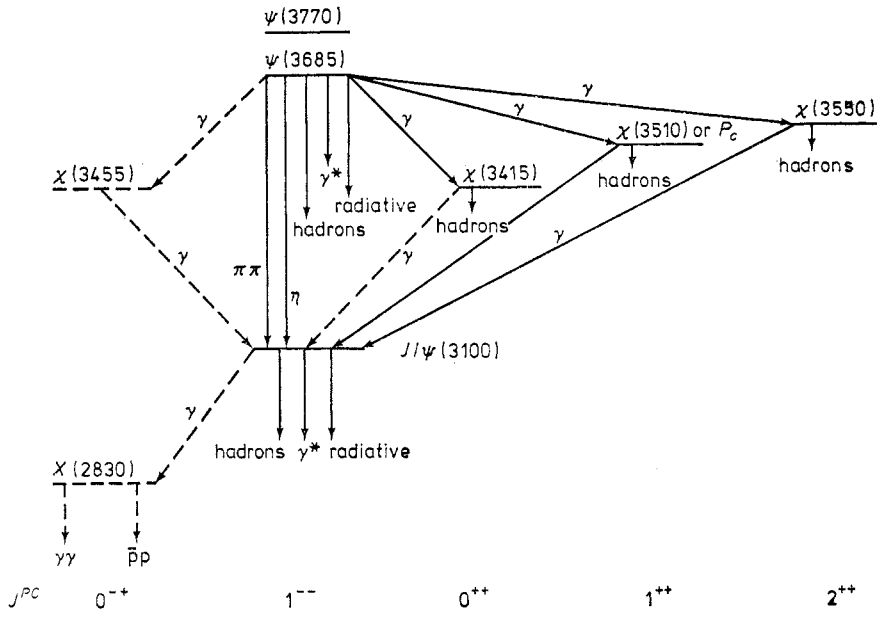

Figure 7. Summary of resonances observed in e^+e^- annihilation in the 2·8–3·8 GeV region. Uncertain states and transitions are indicated by broken lines. J^{PC} quantum number assignments are in some cases tentative, but all are at least consistent with experiment. The notation γ^* refers to decay processes involving intermediate virtual photons, including decays to e^+e^- and $\mu^+\mu^-$. The figure is adapted from Trippe *et al* (1977), with states at 4·42 and 4·03 GeV omitted, but the newly discovered state at 3·77 GeV referred to in the text included.

The hypothesis (B_2) arises in part from the idea that if the binding force comes from the exchange of massless vector gluons, then analogy with QED suggests that V_{spin} is of the order of $(v/c)^2$ relative to V_{si}, at least at short distances where the one-gluon-exchange approximation is supposed to hold. Indications from lattice gauge theories (Wilson 1974, Kogut and Susskind 1974) have been used in support of the idea that at large separations spin-dependent forces fall off rapidly (De Rújula *et al* 1975).

What about the quantitative aspects of the assumptions (B_2) and (B_3)? First, the good news. Study of the hadronic decays of the narrow resonances, which in QCD proceed by $c\bar{c}$ annihilation into virtual gluons, suggest that α_s can be regarded as small ($\sim 0{\cdot}2$). If one neglects spin-dependent forces altogether and drops the r^{-1} term in V_{si} one may solve the resulting Schrödinger equation for the nS states exactly, as it turns out, in terms of Airy functions, i.e. Bessel functions of order $\frac{1}{3}$. Since in the non-relativistic limit the amplitude for, e.g. $\psi \to e^+e^-$, may be expressed in terms of α, m_ψ and ϕ_{1S} ($\boldsymbol{r}=0$), one may determine the three parameters m_c, a and b by fitting, say m_ψ, $m_{\psi'}$ and $\Gamma(\psi \to e^+e^-)$. A typical result of such an exercise is $m_c \sim 1{\cdot}6$–2 GeV. The resulting prediction for $\Gamma(\psi' \to e^+e^-)$ is within a factor of two or so of the experimental value of 2·1 keV. Not too bad, considering the simplicity of the model. Furthermore, the first excited P states are predicted to be at about $\frac{2}{3}$ of the way between ψ and ψ', i.e. at around 3500 MeV, right in the region of the χ states. The inclusion of the r^{-1} term changes these numbers only slightly.

Now for the bad news. (i) If one uses the radial wavefunctions given by the model to compute the widths for the E1 decays of $\psi'(\psi' \to \chi_J + \gamma)$ one gets numbers which are

larger than the experimental ones by a factor of two or more. (ii) If (B_2) holds, $m(^3S_1)-m(^1S_0)$ ought to be *small* compared to $m_{\psi'}-m_{\psi}\approx 590$ MeV. The identification of either $X(2{\cdot}83)$ with η_c or $\chi(3{\cdot}45)$ with η_c' yields Δm of the order of 250 MeV—hardly a small fraction of 590 MeV. (iii) If one imagines that V_{spin} is related to V_{si} in the same way that the non-relativistic spin-dependent interactions between two electrons (coming from the reduction of the Breit operator) are related to the Coulomb interaction, then the splitting of the P states must satisfy the bound (Schnitzer 1975):

$$(E(^3P_2)-E(^3P_0))/E(^3P_1)-E(^3P_0))\geqslant 0{\cdot}8 \tag{5.20}$$

whereas the above assignment gives 0·4 for this ratio.

We shall mention later some of the theoretical work which has been carried out to deal with these difficulties. We turn first to a general discussion of bound states of two spin-$\frac{1}{2}$ Dirac particles and radiative transitions among these states. We shall keep the masses arbitrary, which will facilitate comparison with the atomic case. In view of the uncertainties regarding the forces between quarks, we shall also avoid making specific dynamical assumptions as much as possible in the derivation of formulae for the transition amplitudes.

6. Radiative transitions of two-body bound states of spin-$\frac{1}{2}$ particles

6.1. Relativistic wave equation

We consider bound states of Dirac particles '1' and '2'. We shall refer to them as a quark q_1, with mass m_1, and an antiquark $\bar{q}_2$, with mass m_2, but this is not important for most of this section. We adopt a semi-phenomenological point of view and assume that, to a good first approximation, the dynamics of the $\bar{q}_2 q_1$ bound states is describable by a Dirac-like 'no-pair' equation:

$$(H_1(\boldsymbol{p})+H_2(\boldsymbol{p})+\Lambda_{++}V\Lambda_{++})\,\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2)=\mathscr{E}\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2) \tag{6.1}$$

where $H_i(\boldsymbol{p}_i)=\boldsymbol{\alpha}_i\cdot\boldsymbol{p}_i+\beta_i m_i$, $\Lambda_{++}=\Lambda_+^{(1)}\Lambda_+^{(2)}$ with $\Lambda_+^{(1)}=(E_i(\boldsymbol{p}_i)+H_i(\boldsymbol{p}_i))/2E_i(\boldsymbol{p}_i)$, the Casimir positive-energy projection operator, and V is assumed to be Hermitian and invariant under translation, rotation and inversion.

Equation (6.1) may be viewed as a kind of 'middle-brow' starting point for our discussion. It can be regarded as derived, for example, from a static effective quark–quark interaction H_{int} analogous to the Coulomb interaction H_C:

$$H_{\text{int}}=\tfrac{1}{2}\int\!\!\int j(\boldsymbol{x})\,K(\boldsymbol{x}-\boldsymbol{x}')\,j(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}' \tag{6.2}$$

where $j(\boldsymbol{x})$ is a quark density and the kernel $K(\boldsymbol{x}-\boldsymbol{x}')$ is analogous to $(e^2/4\pi)|\boldsymbol{x}-\boldsymbol{x}'|^{-1}$ in (3.6). Many indices have been suppressed in (6.2). The integrand jKj can be thought of as shorthand for the sum:

$$\sum j_{i\alpha}(\boldsymbol{x})\,K_{ik}{}^{\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}')\,j_{k\beta}(\boldsymbol{x}')$$

where $j_{i\alpha}=\bar{q}_i\Gamma_\alpha q_i$ with $i=1,2,\ldots$, denoting different quark flavours and Γ_α is one of the 16 Dirac matrices 1, γ_μ, $\sigma_{\mu\nu}$, $\gamma_\mu\gamma_5$, or γ_5 (colour indices are still suppressed!). The quantity V in (6.1) is then a linear combination of products of Dirac matrices with the $K_{12}{}^{\alpha\beta}$ serving as coefficients. The derivation of (6.1) from (6.2) is completely analogous to that leading to (3.16) and so also, as in (3.19):

$$\Lambda_+(\boldsymbol{p}_1)\,\Psi=\Lambda_+(\boldsymbol{p}_2)\,\Psi=\Psi. \tag{6.3}$$

However, it should be noted that (6.3), is, in fact, a consequence of (6.1) and need not be stipulated separately.

We define relative and CM coordinates $\boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2$ and $\boldsymbol{R}=c_1\boldsymbol{r}_1+c_2\boldsymbol{r}_2$, with $c_i=m_i(m_1+m_2)^{-1}$ and write $\Psi=\exp(\mathrm{i}\boldsymbol{K}.\boldsymbol{R})\,f_K(\boldsymbol{r})$. The internal wavefunction $f_K(\boldsymbol{r})$ then satisfies (6.1) with $\boldsymbol{p}_1\to\boldsymbol{p}+c_1\boldsymbol{K}$, $\boldsymbol{p}_2\to\boldsymbol{p}-c_2\boldsymbol{K}$ and $\boldsymbol{p}=-\mathrm{i}\partial/\partial\boldsymbol{r}$ conjugate to $\boldsymbol{r}$. For $\boldsymbol{K}=0$ we have:

$$H\psi=(H_1(\boldsymbol{p})+H_2(-\boldsymbol{p})+V)\,\psi(\boldsymbol{r})=E\psi(\boldsymbol{r}) \tag{6.4}$$

where E is the mass of the bound state, $\psi(\boldsymbol{r})$ denotes the wavefunction in the rest frame and

$$V=\Lambda_+^{(1)}(\boldsymbol{p})\,\Lambda_+^{(2)}(-\boldsymbol{p})\,V\Lambda_+^{(1)}(\boldsymbol{p})\,\Lambda_+^{(2)}(-\boldsymbol{p}). \tag{6.5}$$

Equation (6.4) should be regarded as symbolic of the corresponding equation for the Fourier-transformed wavefunction $\tilde{\psi}(\boldsymbol{p})$ in momentum space, where the square root and inverses entering $\Lambda_+^{i}(\boldsymbol{p})$ are well-defined, but it is convenient to keep to the coordinate-space notation.

6.2. *Transformation to Pauli-type wavefunctions*

The reduction of (6.4) to large components:

$$\psi^{++}=\beta_1^{(+)}\beta_2^{(+)}\psi \qquad \beta_i^{(\pm)}\equiv\tfrac{1}{2}(1\pm\beta_i) \tag{6.6}$$

is straightforward. Note that (6.3) implies that H_i can be replaced by $E_i=E_i(\boldsymbol{p})$ in (6.4) and that

$$\begin{gathered}\psi=(1+\zeta_1)(1+\zeta_2)\,\psi^{++}\\ \zeta_1=\frac{\boldsymbol{\alpha}_1.\boldsymbol{p}}{m_1+E_1} \qquad \zeta_2=\frac{-\boldsymbol{\alpha}_2.\boldsymbol{p}}{m_2+E_2}.\end{gathered} \tag{6.7}$$

On multiplying (6.4) by $\beta_1^{(+)}\beta_2^{(+)}$ on the left and making use of the relation:

$$\beta_i^{(+)}\Lambda_+^{(i)}=\frac{E_i+m_i}{2E_i}\,\beta_i^{(+)}(1+\zeta_i)$$

one finds that:

$$H^{++}\psi^{++}=E\psi^{++} \tag{6.8}$$

where:

$$H^{++}=E_1+E_2+\frac{E_1+m_1}{2E_1}\frac{E_2+m_2}{2E_2}\,V_{\mathrm{red}} \tag{6.9}$$

with:

$$V_{\mathrm{red}}=\beta_1^{(+)}\beta_2^{(+)}(1+\zeta_1)(1+\zeta_2)\,V(1+\zeta_1)(1+\zeta_2)\,\beta_1^{(+)}\beta_2^{(+)}. \tag{6.10}$$

Although V_{red} is Hermitian, H^{++} is not. However, on defining a new wavefunction Φ by

$$\psi^{++}=A\Phi \tag{6.11}$$

where:

$$A\equiv\left(\frac{E_1+m_1}{2E_1}\frac{E_2+m_2}{2E_2}\right)^{1/2} \tag{6.12}$$

one sees that†:

$$H_{\mathrm{eff}}\Phi=E\Phi \tag{6.13}$$

† The analogue of (6.13)–(6.15) for spin-0 constituents has been considered from a general point of view by Fong and Sucher (1964) and applied to bound states by Son and Sucher (1967); see also Nowak *et al* (1977).

where H_{eff} is a Hermitian 'effective Hamiltonian' defined by:

$$H_{\text{eff}} = E_1 + E_2 + V_{\text{eff}} \tag{6.14}$$

with

$$V_{\text{eff}} = AV_{\text{red}}A. \tag{6.15}$$

The Hermiticity of H_{eff} is a consequence of the fact that the overall *mapping* from the space of spinors ψ which satisfy the $\Lambda_+{}^i\psi = \psi$ to the spinors Φ which satisfy $\beta_i{}^{(+)}\Phi = \Phi$ is unitary. Thus if one writes:

$$\psi = U\Phi \qquad U = (1+\zeta_1)(1+\zeta_2)\,\beta_1{}^{(+)}\beta_2{}^{(+)}A \tag{6.16}$$

one has $U^\dagger U = \beta_1{}^{(+)}\beta_2{}^{(+)}$ so that $\langle\psi_1|\psi_2\rangle = \langle\Phi_1|\Phi_2\rangle$. We have proceeded in two steps in order to be able to make contact with earlier work (Feinberg and Sucher 1975a). (Note that the *matrix* U is not unitary!).

6.3. Effective potential

To simplify the form of the effective potential V_{eff} we note that a rather general form of a parity-conserving V can be written in terms of Dirac matrices β, ρ and $\boldsymbol{\sigma} = \rho\boldsymbol{\alpha}$ as a sum:

$$V = V^{(1)} + \beta_1\beta_2 V^{(2)} + \rho_1\rho_2 V^{(3)} + \rho_1\rho_2\beta_1\beta_2 V^{(4)} \tag{6.17}$$

where

$$V^{(a)} = U^{(a)} + \sigma_{1i}\sigma_{2j}U_{ij}{}^{(a)} + \sigma_{1i}U_{1;i}{}^{(a)} + \sigma_{2j}U_{2;j}{}^{(a)} \tag{6.18}$$

and the U are rotational scalars or tensors which act only on the spatial variable $\boldsymbol{r}$. To see the non-relativistic limit of (6.14) we shall expand V_{red} in powers of $\boldsymbol{p}/m$. Patient algebra yields:

$$V_{\text{red}} = V_0 + V_{\text{red}}' + \ldots \tag{6.19}$$

where

$$V_0 = V^{(1)} + V^{(2)} \tag{6.20}$$

$$V_{\text{red}}' = \boldsymbol{\sigma}_1\cdot\boldsymbol{p}_1(V^{(1)} - V^{(2)})\,\boldsymbol{\sigma}_1\cdot\boldsymbol{p}_1/4m_1{}^2 + (1\leftrightarrow 2) + [\boldsymbol{\sigma}_1\cdot\boldsymbol{p}_1,\,[\boldsymbol{\sigma}_2\cdot\boldsymbol{p}_2,\,V^{(3)}]_+]_+/4m_1m_2 + [\boldsymbol{\sigma}_1\cdot\boldsymbol{p}_1,\,[\boldsymbol{\sigma}_2\cdot\boldsymbol{p}_2,\,V^{(4)}]_-]_-/4m_1m_2 \tag{6.21}$$

with $\boldsymbol{p}_1 = -\boldsymbol{p}_2 = \boldsymbol{p}$ in our case. Terms of the type $V^{(3)}$ and $V^{(4)}$, if at all present in (6.17), will be assumed not to exceed the $V^{(1)}$ or $V^{(2)}$ type of term in order of magnitude; then the scale is set by V_0 and V_{red}' is of the relative order of v^2/c^2 while the unwritten remainder in (6.19) is of the relative order of v^4/c^4 or higher.

If V is regarded as coming from (6.2) then $U_{n;i}{}^{(a)} = 0$ and the other U in (6.18) will be local: $U^{(a)} = U^{(a)}(r)$ and $U_{ij}{}^{(a)} = \delta_{ij}A^{(a)}(r) - 3\hat{r}_i\hat{r}_jB^{(a)}(r)$. For simplicity, we shall assume that this is the case at least for $a=1$ and $a=2$. Then we can rewrite V_0 in the form:

$$V_0 = U_0(r) + \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 U_{\text{ss}}(r) + \mathscr{S}U_{\text{T}}(r) \tag{6.22}$$

with spin–orbit terms contained in (6.21). In (6.22):

$$U_0(r) = U^{(1)}(r) + U^{(2)}(r) \tag{6.23}$$

is a central interaction and $U_{\text{ss}} = (A^{(1)} - B^{(1)}) + (A^{(2)} - B^{(2)})$ and $U_{\text{T}} = B^{(1)} + B^{(2)}$ represent spin–spin and tensor interaction respectively, with $\mathscr{S} \equiv \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - 3\boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{r}}\,\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{r}}$ the familiar tensor operator.

On expanding the E_i in (6.14) in powers of p^2/m^2 the non-relativistic limit of (6.13) is seen to have the form:

$$H_{\rm nr}\phi=\left(\frac{p^2}{2\mu}+V_0\right)\phi=W\phi \tag{6.24}$$

where $\mu=m_1m_2(m_1+m_2)^{-1}$ is the reduced mass and ϕ, a Pauli-type wavefunction, is the non-relativistic limit of Φ as well as of ψ^{++} and ψ. The level shift $\delta W=E-(m_1+m_2+W)$ arising from the neglected terms is given to the lowest order in v^2/c^2 by $\delta W=\langle\phi\,|\,H_{\rm nr}'\,|\,\phi\rangle$ where

$$H_{\rm nr}'=\frac{-(p^2)^2}{8m_1{}^3}\,\frac{-(p^2)^2}{8m_2{}^3}-\left[\frac{p^2}{4K^2},\,V_0\right]_+ +V_{\rm red}' \tag{6.25}$$

with $K^2=2m_1{}^2m_2{}^2(m_1{}^2+m_2{}^2)^{-1}$ and $V_{\rm red}'$ defined by (6.21).

It will also be useful in the next section to consider the case of *all* spin-dependent forces being small, i.e. with both $U_{\rm ss}$ and $U_{\rm T}$ in (6.22) of the order of v^2/c^2 relative to U_0, and therefore to define a Pauli wavefunction $\tilde{\phi}(\boldsymbol{r})$ satisfying:

$$\left(\frac{p^2}{2\mu}+U_0(r)\right)\tilde{\phi}(\boldsymbol{r})=\tilde{W}\tilde{\phi}(\boldsymbol{r}). \tag{6.26}$$

The form of the two-body interaction given by (6.17) and (6.18) is sufficiently general to include almost all the types of potential which arise from the exchange of a spin-0 or spin-1 boson between the quarks. Not included are terms which arise, for example, from emission of a vector boson B by the Dirac moment of '1' followed by absorption of B via a Pauli moment of '2'. A detailed discussion of the potential arising from exchange of a single boson between the quarks may be found in Sucher (1978).

6.4. *Theory of radiative decay*

Within the rather general framework just outlined the amplitude for the transition $\Psi_{\rm I}\to\Psi_{\rm F}+\gamma$ is given by $-e/\sqrt{2k}\mathcal{M}$ where, in analogy with (3.22) and (3.23):

$$\mathcal{M}=\mathcal{M}_{\rm dir}+\mathcal{M}_{\rm pair} \tag{6.27}$$

is the sum of a 'direct term' $\mathcal{M}_{\rm dir}$ and a 'pair term' $\mathcal{M}_{\rm pair}$ defined by:

$$\mathcal{M}_{\rm dir}=\langle\Psi_{\rm F}\,|\,Q_1\boldsymbol{\alpha}_1\cdot\boldsymbol{\epsilon}^*\eta_1+Q_2\boldsymbol{\alpha}_2\cdot\boldsymbol{\epsilon}^*\eta_2\,|\,\Psi_{\rm I}\rangle \tag{6.28}$$

$$\mathcal{M}_{\rm pair}\simeq\left\langle\Psi_{\rm F}\,\middle|\,Q_1\left(\boldsymbol{\alpha}_1\cdot\boldsymbol{\epsilon}^*\eta_1\Lambda_-{}^{(1)}\frac{V}{2m_1}+\frac{V}{2m_1}\,\Lambda_-{}^{(1)}\boldsymbol{\alpha}_1\cdot\boldsymbol{\epsilon}^*\eta_1\right)\,\middle|\,\Psi_{\rm I}\right\rangle+(1\leftrightarrow2). \tag{6.29}$$

Here eQ_1 and eQ_2 is the charge of q_1 and $\bar{q}_2$ respectively and $\eta_i=\exp(-i\boldsymbol{k}\cdot\boldsymbol{r}_i)$. The time-ordered diagrams corresponding to (6.21) and (6.22) are analogous to those shown in figures 3(a) and 4(b) for the two-electron problem, with the broken line representing the interaction V rather than the Coulomb interaction.

6.4.1. The direct term. The amplitude $\mathcal{M}_{\rm dir}$ can be expressed in terms of the large components Ψ^{++} satisfying $\beta_i\Psi^{++}=\Psi^{++}$ by using (5.13) to write

$$\Psi=(1+\xi_1)(1+\xi_2)\,\Psi^{++}$$

with $\xi_i = \boldsymbol{\alpha}_i . \boldsymbol{p}_i (m_i + E_i(\boldsymbol{p}_i))^{-1}$ and substituting this form for Ψ into (6.28). A short calculation yields:

$$\mathcal{M}_{\mathrm{dir}} = Q_1 \left\langle \Psi_{\mathrm{F}}{}^{++} \left| \eta_1 (P_1{}^{(+)} + \mathrm{i}\boldsymbol{\sigma}_1 \times \boldsymbol{P}_1{}^{(-)}) . \boldsymbol{\epsilon}^* \left(\frac{2E_2}{E_2 + m_2} \right) \right| \Psi_{\mathrm{I}}{}^{++} \right\rangle + (1 \leftrightarrow 2) \quad (6.30)$$

where

$$\boldsymbol{P}_1{}^{(\pm)} \equiv \eta_1{}^{-1} \left[\frac{\boldsymbol{p}_1}{E_1 + m_1}, \eta_1 \right]_{\pm} \quad (6.31)$$

which may be written more explicitly as:

$$\boldsymbol{P}_1{}^{(\pm)}(\boldsymbol{p}_1, \boldsymbol{k}) = \frac{\boldsymbol{p}_1 + \boldsymbol{k}_1}{E_1(\boldsymbol{p}_1 + \boldsymbol{k}) + m_1} \pm \frac{\boldsymbol{p}_1}{E_1(\boldsymbol{p}_1) + m_1}. \quad (6.32)$$

With the decaying state at rest, $\Psi_{\mathrm{I}} = \psi_{\mathrm{I}}(\boldsymbol{r})$ and if we neglect the dependence of $f_{-k}(\boldsymbol{r})$ on $-\boldsymbol{k} = \boldsymbol{K}_{\mathrm{F}}$, $\Psi_{\mathrm{F}} \approx \exp(\mathrm{i}\boldsymbol{K}_{\mathrm{F}} . \boldsymbol{R})\, \psi_{\mathrm{F}}(\boldsymbol{r})$. Then:

$$\mathcal{M}_{\mathrm{dir}} \approx (2\pi)^3\, \delta(\boldsymbol{K}_{\mathrm{F}} + \boldsymbol{k})\, M_{\mathrm{dir}} \quad (6.33)$$

$$M_{\mathrm{dir}} = Q_1 \langle \psi_{\mathrm{F}}{}^{++} | \tilde{\eta}_1 (\boldsymbol{P}_1{}^{(+)} + \mathrm{i}\boldsymbol{\sigma} \times \boldsymbol{P}_1{}^{(-)}) . \boldsymbol{\epsilon}^* A_2(\boldsymbol{p}) | \psi_{\mathrm{I}}{}^{++} \rangle + (1 \leftrightarrow 2) \quad (6.34)$$

where now

$$\boldsymbol{P}_i{}^{(\pm)} = \boldsymbol{P}_i{}^{(\pm)}(\boldsymbol{p}, \boldsymbol{k})$$

and

$$\tilde{\eta}_1 = \exp(-\mathrm{i}c_2 \boldsymbol{k} . \boldsymbol{r}) \qquad \tilde{\eta}_2 = \exp(\mathrm{i}c_1 \boldsymbol{k} . \boldsymbol{r}) \qquad A_i(\boldsymbol{p}) = 2E_i(\boldsymbol{p})(E_i(\boldsymbol{p}) + m_i)^{-1}. \quad (6.35)$$

We shall ultimately be interested in the case where $\tilde{q}_2$ is the antiparticle of q_1. Then CP invariance and the relation $CP = (-1)^{S+1}$, which follows from (5.1), imply that the spin S is an *exact* quantum number, independent of any non-relativistic approximations. Furthermore, for a one-photon transition with no parity change, CP changes its sign and hence $\Delta S = \pm 1$. Under this circumstance the spin-independent 'electric' term $\boldsymbol{P}_1{}^{(+)}$ in (6.34) gives zero contribution. (Of course, if the parity changes this term normally gives the leading contribution to an allowed E1 transition.) With this in mind we concentrate on the 'magnetic' term:

$$M_{\mathrm{dir}}{}^{\mathrm{mag}} = \mathrm{i}Q_1 \langle \psi_{\mathrm{F}}{}^{++} | \tilde{\eta}_1 \boldsymbol{\sigma} \times \boldsymbol{P}_1{}^{(-)} . \boldsymbol{\epsilon}^* A_2(\boldsymbol{p}) | \psi_{\mathrm{I}}{}^{++} \rangle + (1 \leftrightarrow 2). \quad (6.36)$$

So far *no* expansion in powers of v/c has been made and (6.36) may be useful in a more general context. However, we shall now assume that an expansion in v/c makes sense and find the leading terms in this expansion.

6.4.2. Expansion in v/c. From (6.32) and (6.35) we get, counting k/m as of the order of v^2/c^2, and kr as of the order of v/c:

$$\boldsymbol{P}_1{}^{(-)}(\boldsymbol{p}, \boldsymbol{k}) = -\boldsymbol{k}/2m_1 + (\boldsymbol{p}^2 \boldsymbol{k} + 2\boldsymbol{p} . \boldsymbol{k}\boldsymbol{p})/8m_1{}^3 + \mathcal{O}(v^5/c^5)$$

$$\tilde{\eta}_1 = 1 - \mathrm{i}c_2 (\boldsymbol{k} . \boldsymbol{r}) - c_2{}^2 (\boldsymbol{k} . \boldsymbol{r})^2/2 + \mathcal{O}(v^3/c^3). \quad (6.37)$$

On writing $\tilde{\eta}_1 = 1 + (\tilde{\eta}_1 - 1)$ in (6.36) and substituting $\psi^{++} = A\Phi$ from (6.11), the '1' term simplifies to:

$$\mathrm{i}Q_1 \langle \Phi_{\mathrm{F}} | \boldsymbol{\sigma}_1 . \boldsymbol{P}_1{}^{(-)} \times \boldsymbol{\epsilon}^* A_1{}^{-1}(\boldsymbol{p}) | \Phi_{\mathrm{I}} \rangle + (1 \leftrightarrow 2). \quad (6.38)$$

In the '$\tilde{\eta}_1 - 1$' term, $\boldsymbol{P}_1^{(-)}$ may here be replaced by $-\boldsymbol{k}/2m$, A and $A_2(\boldsymbol{p})$ each by unity to give a term:

$$\frac{iQ_1}{2m_1}\langle\phi_F|\boldsymbol{\sigma}_1.\boldsymbol{k}\times\boldsymbol{\epsilon}^*(1-\tilde{\eta}_1)|\phi_I\rangle+(1\leftrightarrow 2). \tag{6.39}$$

Since $A_1^{-1}(\boldsymbol{p})\approx 1-\boldsymbol{p}^2/4m_1^2$, (6.38) reduces to:

$$iQ_1\left\langle\Phi_F\left|\boldsymbol{\sigma}_1.\left(\frac{-\boldsymbol{k}}{2m_1}+\frac{\boldsymbol{p}^2\boldsymbol{k}+\boldsymbol{p}.\boldsymbol{k}\boldsymbol{p}}{4m_1{}^3}\right)\times\boldsymbol{\epsilon}^*\right|\Phi_I\right\rangle+\ldots. \tag{6.40}$$

Combining terms we see that:

$$M_{dir}{}^{mag}=M_{dir}{}'+M_{dir}{}''+\ldots \tag{6.41}$$

where

$$M_{dir}{}'=-i\frac{Q_1}{2}\langle\Phi_F|\Sigma_1|\Phi_I\rangle+(1\leftrightarrow 2) \tag{6.42}$$

$$M_{dir}{}''=+\frac{iQ_1}{2m_1}\langle\Phi_F|\boldsymbol{\sigma}_1.[(1-\tilde{\eta}_1)\,\boldsymbol{k}+(\boldsymbol{p}^2\boldsymbol{k}+\boldsymbol{p}.\boldsymbol{k}\boldsymbol{p})/2m_1{}^2]\times\boldsymbol{\epsilon}^*|\Phi_I\rangle+(1\leftrightarrow 2) \tag{6.43}$$

and

$$\Sigma_1=\boldsymbol{\sigma}_1.\boldsymbol{k}\times\boldsymbol{\epsilon}^*/m_1. \tag{6.44}$$

The quantity $M_{dir}{}''$ is explicitly of the order of v^4/c^4 and so the wavefunctions Φ_F and Φ_I which satisfy (6.13) may be immediately replaced by the non-relativistic limits ϕ_I and ϕ_F which satisfy (6.24). The nominally leading term $M_{dir}{}'$ requires more care in its evaluation.

6.4.2.1. Discussion of $M_{dir}{}'$. The expression (6.42) for $M_{dir}{}'$ seems to be very similar to (2.11) so that $M_{dir}{}'$ might well be expected to be small. However, there is an important difference from the case of, say, the He-like ion where only the analogue of $M_{dir}{}''$ played an important role in M1 transitions. This difference arises from the fact that we are interested, for charmonium, in the case $Q_2=-Q_1$. Indeed, with $Q_2=+Q_1$ (6.42) reduces, with $m_1=m_2$ and $\boldsymbol{S}\equiv(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)/2$, to:

$$\frac{-iQ_1}{m_1}\langle\Phi_F|\boldsymbol{S}.\boldsymbol{k}\times\boldsymbol{\epsilon}^*|\Phi_I\rangle \tag{6.45}$$

which vanishes, since either Φ_I or Φ_F is a singlet state. With $Q_2=-Q_1$, $M_{dir}{}'$ is in general non-vanishing. However, if spin-dependent forces are small $M_{dir}{}'$ will also be small, because of the approximate orthogonality of the radial wavefunctions.

6.4.3. $J=1\leftrightarrow J=0$ transitions. To discuss $M_{dir}{}^{mag}$ further let us confine our attention to transitions between a $J=1$ and $J=0$ state of the same parity. Without any approximation we may write:

$$\Phi_{J=0}(\boldsymbol{r})=|{}^1S_0\rangle \qquad \Phi_{J=1}(\boldsymbol{r})=a_0|{}^3S_1\rangle+a_2|{}^3D_1\rangle \tag{6.46}$$

where $|a_0|^2+|a_2|^2=1$ and

$$|{}^1S_0\rangle=\Phi^{(s)}(r)\,\chi_0/\sqrt{4\pi} \qquad |{}^3S_1\rangle=\Phi^{(t)}(r)\,\chi_1{}^m/\sqrt{4\pi} \tag{6.47}$$

with χ_0 a singlet and $\chi_1{}^m$ a triplet Pauli spin wavefunction. Since $[\boldsymbol{l}^2,\Sigma_1]=0$ implies

$\langle {}^1S_0|\Sigma_1|{}^3D_2\rangle = 0$ we get, on substituting (6.46) into (6.42) and (6.43) with $J_I = 1$ and $J_F = 0$, say:

$$M_{dir}' = \frac{iQ_1 a_0}{2}\langle\chi_F|\Sigma_1|\chi_I\rangle I_{dir}' + (1\to 2) \tag{6.48}$$

$$I_{dir}' = -\langle\Phi_F^{(s)}(r)|\Phi_I^{(t)}(r)\rangle$$

and, on integrating over angles in the pure S-state term:

$$M_{dir}'' = \frac{iQ_1 a_0}{2}\langle\chi_F|\Sigma_1|\chi_I\rangle I_{dir}'' + (1\leftrightarrow 2) + (M_{dir}'')_D \tag{6.49}$$

where the last term denotes the D-state contribution and

$$I_{dir}'' = \langle\Phi_F^{(s)}(r)|(1 - j_0(c_2 kr)) + 2p^2/3m_1^2|\Phi_I^{(t)}(r)\rangle. \tag{6.50}$$

If the spin-dependent terms in the Hamiltonian H_{eff} defined by (6.14) can be neglected, then there is no mixing between ${}^3S_1$ and ${}^3D_1$ (absence of tensor force) and the radial S-state wavefunctions $\Phi^{(s)}(r)$ and $\Phi^{(t)}(r)$ are eigenfunctions of the same operator (absence of spin–spin interaction). Then, for a transition between S states with principal quantum numbers n_F and n_I:

$$\langle\Phi_F(r)|\Phi_I(r)\rangle = \delta_{n_F n_I} \tag{6.51}$$

and keeping $k \neq 0$ even if $n_F = n_I$:

$$M_{dir}' \to \frac{-iQ_1}{2}\langle\chi_F|\Sigma_1|\chi_I\rangle\,\delta_{n_F n_I} + (1\leftrightarrow 2). \tag{6.52}$$

On also replacing the Φ by the $\tilde{\phi}$ defined by (6.26) we get, to leading order in v/c:

$$M_{dir}'' \approx \frac{iQ_1}{2}\langle\chi_F|\Sigma_1|\chi_I\rangle\left\langle\tilde{\phi}_F\left|\frac{c_2^2 k^2 r^2}{6} + \frac{2}{3}\frac{\boldsymbol{p}^2}{m_1^2}\right|\tilde{\phi}_I\right\rangle + (1\leftrightarrow 2). \tag{6.53}$$

As a check, consider a transition with $n_F \neq n_I$ and let $m_2 \to \infty$ so that $c_2 \to 1$. Then:

$$M_{dir}^{mag} \to \frac{iQ_1}{2}\left\langle\chi_F\left|\frac{\boldsymbol{\sigma_1}.\boldsymbol{k}\times\boldsymbol{\epsilon}^*}{m_1}\right|\chi_I\right\rangle\left\langle\tilde{\phi}_F\left|\frac{k^2 r^2}{6} + \frac{2}{3}\frac{\boldsymbol{p}^2}{m_1^2}\right|\tilde{\phi}_I\right\rangle. \tag{6.54}$$

For $Q_1 = -1$ this agrees, up to a sign arising from a change in conventions, with (3.25), derived for the direct amplitude $M^{(1)}$ for magnetic dipole transitions between S states in H-like ions, with Σ_{fi} defined by (2.5).

Let us specialise (6.53) to the case where '2' is the antiparticle of '1', relevant for the charmonium model:

$$m_1 = m_2 = m \qquad Q_1 = -Q_2 = Q. \tag{6.55}$$

Since one of the two states χ_F and χ_I is symmetric and the other antisymmetric on interchange of '1' and '2', we have:

$$\langle\chi_F|\boldsymbol{\sigma}_2|\chi_I\rangle = -\langle\chi_F|\boldsymbol{\sigma}_1|\chi_I\rangle. \tag{6.56}$$

Then (6.53) reduces to:

$$M_{dir}'' = iQ\left\langle\chi_F\left|\frac{\boldsymbol{\sigma_1}.\boldsymbol{k}\times\boldsymbol{\epsilon}^*}{m}\right|\chi_I\right\rangle\left\langle\phi_F(r)\left|\frac{1}{24}k^2 r^2 + \frac{2}{3}\frac{\boldsymbol{p}^2}{m^2}\right|\phi_I(r)\right\rangle. \tag{6.57}$$

The inclusion of spin dependence in H_{eff} leads to an I_{dir}' which is no longer zero

when $n_F \neq n_I$. To discuss this in a general way we note that H_{eff} may always be written in the form:

$$H_{\text{eff}} = H_{\text{eff}}^{\text{si}} + H_{\text{eff}}^{\text{ss}} + H_{\text{eff}}^{\text{T}} + H_{\text{eff}}^{\text{so}} \tag{6.58}$$

where $H_{\text{eff}}^{\text{si}}$ is the spin-independent part of H_{eff}, and the remaining terms are spin–spin, tensor and spin–orbit interactions, which may be written in the form:

$$H_{\text{eff}}^{\text{ss}} = U_{\text{eff}}^{\text{ss}} \boldsymbol{\sigma}_1 . \boldsymbol{\sigma}_2 \qquad H_{\text{eff}}^{\text{T}} = U_{\text{eff}}^{\text{T}} \qquad H_{\text{eff}}^{\text{so}} = U_{\text{eff}}^{\text{so}} . \boldsymbol{l} \tag{6.59}$$

where $U_{\text{eff}}^{\text{so}} = U_{\text{eff};1}^{\text{so}} \boldsymbol{\sigma}_1 + U_{\text{eff};2}^{\text{so}} \boldsymbol{\sigma}_2$ and all U act only on spatial coordinates (but need not be local). The singlet radial wavefunction $\Phi^{(s)}(r)$ then satisfies:

$$(H_{\text{eff}}^{\text{si}} - 3U_{\text{eff}}^{\text{ss}})\, \Phi^{(s)}(r) = E_s \Phi^{(s)}(r). \tag{6.60}$$

If one neglects the effects on the $^3S_1$ radial wavefunction $\Phi^{(t)}(r)$ arising from mixing with the $^3D_1$ state, one also has:

$$(H_{\text{eff}}^{\text{si}} + U_{\text{eff}}^{\text{ss}})\, \Phi^{(t)}(r) = E_t \Phi^{(t)}(r). \tag{6.61}$$

On taking the scalar product of (6.61) with $\Phi^{(s)}$ and making use of (6.60) one gets:

$$\langle \Phi^{(s)} | \Phi^{(t)} \rangle = -4 \frac{\langle \Phi^{(s)} | U_{\text{eff}}^{\text{ss}} | \Phi^{(t)} \rangle}{E_s - E_t} \tag{6.62}$$

and the same equation holds with $-4 \to +4$ and 's' and 't' interchanged. If all spin-dependent forces are of the order of v^2/c^2 relative to $H_{\text{eff}}^{\text{si}}$ then (6.62) is of the order of v^2/c^2 and the correction from mixing is of higher order. On replacing the Φ by the non-relativistic $\tilde{\phi}$ of (6.26) one therefore finds, for a transition between S states with $n_F \neq n_I$:

$$I_{\text{dir}}' \approx \mp 4 \frac{\langle \tilde{\phi}_F(r) | U_{\text{eff}}^{\text{ss}} | \tilde{\phi}_I(r) \rangle}{\tilde{W}_I - \tilde{W}_F}. \tag{6.63}$$

6.4.4. The pair term. The leading term in $\mathcal{M}_{\text{pair}}$, defined by (6.29), can be found by expressing Ψ in terms of Ψ^{++} and approximating E_i by m_i everywhere, so that also $\Lambda_-^{(i)}(\boldsymbol{p}_i) \to \frac{1}{2}(1 - \beta_i) - \boldsymbol{\alpha}_i . \boldsymbol{p}_i / 2m_i$. The result may be written in the relatively compact form:

$$\mathcal{M}_{\text{pair}} \approx \frac{Q_1}{2m_1} \left\langle \Psi_F^{++} \middle| \left[\boldsymbol{\alpha}_1 . \boldsymbol{\epsilon}^* \eta_1, \left[V, \frac{\boldsymbol{\alpha}_1 . \boldsymbol{p}_1}{2m_1} \right]_- \right]_- + \left[\frac{\boldsymbol{\alpha}_2 . \boldsymbol{p}_2}{2m_2}, [V, \boldsymbol{\alpha}_1 . \boldsymbol{\epsilon}^* \eta_1]_+ \right]_+ \middle| \Psi_I^{++} \right\rangle + (1 \leftrightarrow 2). \tag{6.64}$$

For a transition in which the parity changes, (6.64) just represents, with η_1 replaced by unity, a relativistic correction to, for example, an E1 amplitude computed in a non-relativistic approximation. For a transition in which the parity does not change, η_1 may be replaced by $\eta_1 - 1 \approx -i\boldsymbol{k} . \boldsymbol{r}_1$ and (6.64) represents a contribution $\mathcal{M}_{\text{pair}}^{\text{mag}}$ of the same order of magnitude as the direct terms discussed above.

As an illustration, let us apply (6.64) to the special case of a potential V which is diagonal in Dirac matrices, i.e. consider the contribution of only the first term $U^{(1)}$ in the eight terms contained in the general form (6.17) for V. Then the anticommutator term in (6.64) vanishes and with $U^{(1)} \equiv U_v(r)$ local the first term gives, in the CM frame:

$$M_{\text{pair}}^{\text{mag}} \approx \frac{-Q_1}{2m_1} \langle \psi_F^{++} | \tilde{\eta}_1 \boldsymbol{\sigma}_1 . \boldsymbol{\epsilon}^* \times \hat{r} U_v'(r) | \psi_I^{++} \rangle + (1 \leftrightarrow 2). \tag{6.65}$$

For a transition between a $^3S_1$ and $^1S_0$ state, with the D state neglected, $\tilde{\eta}_1\hat{r}$ may be replaced by its spherical average $-ij_1(c_2kr)\,\hat{k}$. If one also replaces $j_1(x)$ by its leading term $\frac{1}{3}x$, and the wavefunctions ψ^{++} by their non-relativistic limits one gets to leading order in v/c:

$$M_{\text{pair}}^{\text{mag}} = \frac{iQ_1}{2}\langle\chi_F|\Sigma_1|\chi_I\rangle\left\langle\tilde{\phi}_F(r)\left|-\frac{c_2 r U_v'(r)}{3m_1}\right|\tilde{\phi}_I(r)\right\rangle + (1\leftrightarrow 2). \tag{6.66}$$

If $Q=-1$ and $m_2\to\infty$ then $c_2\to 1$ and (6.66) coincides (again up to a sign) with the pair term (3.29) discussed in the hydrogenic case.

For later use we consider the more general case of a potential V of the form:

$$V = X_s(r)\,\beta_1\beta_2 + X_v(r) + Y_v(r)\,\boldsymbol{\alpha}_1.\boldsymbol{\alpha}_2 + Z_v(r)\,\boldsymbol{\alpha}_1.\hat{r}\boldsymbol{\alpha}_2.\hat{r}. \tag{6.67}$$

The first term is of the kind which would arise from exchange of a scalar meson between quarks and the rest are of the type coming from vector meson exchange. Notice that only the second line in (6.64) will contribute in the case of operators such as the last two terms in (6.67), which are 'odd', i.e. anticommute with the β_i, whereas only the first line contributes for even operators such as the first two terms in (6.67). On substitution of (6.67) into (6.64) and specialisation to $^3S_1\to{}^1S_0+\gamma$ transitions, with neglect of mixing with the $^3D_1$ state one gets, to leading order in v/c:

$$M_{\text{pair}}^{\text{mag}} \approx \frac{iQ_1}{2}\langle\chi_F|\Sigma_1|\chi_I\rangle\, I_{\text{pair}} + (1\leftrightarrow 2) \tag{6.68}$$

where

$$I_{\text{pair}} = \langle\tilde{\phi}_I(r)|\,X_s/m_1 - c_2(rX_v' + rY_v' - Z_v)/3m_1\,|\tilde{\phi}_F(r)\rangle. \tag{6.69}$$

6.5. *Partial summary and cases of special interest*

In the preceding subsections a number of formulae of varying degrees of exactness have been derived for one-photon radiative transition amplitudes for two-body bound states, described in the CM system by a $4\times 4=16$ component Dirac-type wavefunction $\psi(\boldsymbol{r})$ satisfying (6.4). All amplitudes can be expressed in terms of the Pauli-type wavefunctions $\Phi(\boldsymbol{r})$ which are eigenfunctions of the Hamiltonian H_{eff} defined by (6.14) and have, in the standard representation of the β_i matrices, only $2\times 2=4$ non-vanishing components; the symbol ψ^{++} entering these formulae should simply be regarded as an abbreviation for $A\Phi(r)$ where A is defined by (6.12). To leading order in v^2/c^2 the relevant amplitudes can further be expressed as matrix elements of simple operators taken between eigenfunctions $\phi(\boldsymbol{r})$ of H_{nr}, the non-relativistic limit of H_{eff} defined by (6.22) and (6.24).

The formulae become especially simple when the spin-dependent part of H_{eff} is small, of the order of v^2/c^2 relative to the spin-independent part $H_{\text{eff}}^{\text{si}}$, in which case all amplitudes can be expressed in terms of solutions $\tilde{\phi}(\boldsymbol{r})$ of (6.26):

$$\left(\frac{\boldsymbol{p}^2}{2\mu} + U_0\right)\tilde{\phi}(\boldsymbol{r}) = \tilde{W}\tilde{\phi}(\boldsymbol{r}) \tag{6.70}$$

where U_0 is the leading term in $H_{\text{eff}}^{\text{si}}$. For the particular case of a $^3S_1\to{}^1S_0+\gamma$ or $^1S_0\to{}^3S_1+\gamma$ transition between states with principal quantum numbers n_F and n_I

the result for the decay amplitude may be written in the form:

$$M=\frac{iQ_1}{2}\langle\chi_F|\Sigma_1|\chi_I\rangle(I_{dir}'+I_{ret}+I_{kin}+I_{pair})+(1\leftrightarrow 2) \tag{6.71}$$

where

$$I_{dir}'=I_{spin} \qquad (n_F\neq n_I) \tag{6.72}$$
$$=-1 \qquad (n_F=n_I)$$

with

$$I_{spin}=\mp 4\langle\tilde{\phi}_F(r)|U_{ss}^{tot}(r)|\tilde{\phi}_I(r)\rangle/(\tilde{W}_I-\tilde{W}_F) \qquad (n_F\neq n_I) \tag{6.73}$$

$$I_{ret}=\langle\tilde{\phi}_F(r)|c_2^2k^2r^2/6|\tilde{\phi}_I(r)\rangle \tag{6.74}$$

$$I_{kin}=\langle\tilde{\phi}_F(r)|2\boldsymbol{p}^2/3L^2|\tilde{\phi}_I(r)\rangle \tag{6.75}$$

representing spin, retardation and kinetic effects, respectively. Here $c_i=m_i(m_1+m_2)^{-1}$, $\Sigma_i=\boldsymbol{\sigma}_I.\boldsymbol{k}\times\boldsymbol{\epsilon}^*/m_i$, $L^2=m_1^2m_1^2(m_1+m_2)(m_1^3+m_2^3)^{-1}$ and $\boldsymbol{\sigma}_1.\boldsymbol{\sigma}_2U_{ss}^{tot}(r)$ is now defined as the leading spin–spin term in H_{eff}. In (6.73) the upper sign is to be used for $J_I=1$, $J_F=0$ and the lower sign for $J_I=0$, $J_F=1$.

The explicit expressions for U_0, $U_{ss}^{tot}(r)$ and the pair term I_{pair} depend on the form of V. For the case where the general potential is restricted to the form (6.67), U_0 in (6.70) is given by:

$$U_0=X_s(r)+X_v(r) \tag{6.76}$$

U_{ss}^{tot} in (6.73) by:

$$U_{ss}^{tot}(r)=(6m_1m_2)^{-1}[Z_v/r^2+3Z_v'/r+Z_v''-\nabla^2(Y_v+Z_v)] \tag{6.77}$$

and I_{pair} is given by (6.69). For a local potential such as (6.76), one may use the relations:

$$\langle(\Delta\tilde{W})^2r^2\rangle=\langle[H_{nr},[H_{nr},r^2]_-]_-\rangle=\langle 2rU_0'-2\boldsymbol{p}^2/\mu\rangle \tag{6.78}$$

$$\langle\boldsymbol{p}^2/2\mu\rangle=-\langle U_0\rangle \tag{6.79}$$

where the bracket denotes a matrix element taken between orthogonal $\tilde{\phi}$, to express I_{ret} and I_{kin} in terms of U_0 and U_0', which is sometimes useful for computational purposes. (Note that the identity (6.78) just gives the virial theorem when $\tilde{\phi}_F=\tilde{\phi}_I$.)

For the case of equal masses ($m_1=m_2=m_c$) and opposite charges ($Q_1=-Q_2=Q$) of interest for charmonium, we may write, using (6.71):

$$M=iQ\langle\chi_F|\boldsymbol{\sigma}_1.\boldsymbol{k}\times\boldsymbol{\epsilon}^*/m_c|\chi_I\rangle\, I_{tot} \tag{6.80}$$

where

$$I_{tot}=I_{dir}'+I_{ret}+I_{kin}+I_{pair} \tag{6.81}$$

with

$$I_{ret}=\langle\tilde{\phi}_F|k^2r^2/24|\tilde{\phi}_I\rangle \qquad I_{kin}=\langle\tilde{\phi}_F|2\boldsymbol{p}^2/3m_c^2|\tilde{\phi}_I\rangle$$
$$I_{pair}=\langle\tilde{\phi}_F|X_s/6m_c-(rX_v'+rY_v'-Z_v)/6m_c|\tilde{\phi}_I\rangle \tag{6.82}$$

and $m_1=m_2=m_c$ in (6.70) and in I_{dir}', defined by (6.72), (6.73) and (6.77).

The results summarised here are essentially the same as those given previously by Feinberg and Sucher (1975a,b). The only difference in treatment is the introduction of the transformation (6.11) which leads to a Hermitian H_{eff} and simplifies the subsequent discussion. (For the case $m_1=m_2$ and $n_F\neq n_I$ the quantities denoted by I_1, I_2 and I_3 in Feinberg and Sucher (1975a) coincide with an $I_{ret}+I_{kin}$, I_{spin} and I_{pair} respectively, when higher-order retardation effects included in I_1 and I_3 are

dropped, and a missing factor of four is supplied in the definition of I_2; the attempt in that paper to state the results in a form valid even when $m_1 \neq m_2$, without taking up more space, backfired and led instead to garbled formulae which should be ignored when $m_1 \neq m_2$.)

For completeness we also list the leading contributions to the tensor, spin–orbit and spin-independent parts of $V_{\rm red}$ when V has the form (6.67) (Feinberg and Sucher 1975b):

$$\begin{aligned} V_{\rm red}{}^{\rm T} &\approx (12m_1m_2)^{-1}[Y_{\rm v}'/r - Y_{\rm v}'' - 2Z_{\rm v}/r^2 + Z_{\rm v}'/r]\ \mathscr{S} \\ V_{\rm red}{}^{\rm so} &\approx r^{-1}[(X_{\rm v}' - X_{\rm s}')/4m_1{}^2 - (Y_{\rm v}' - r^{-1}Z_{\rm v})/2m_1m_2]\ \boldsymbol{\sigma}_1 . \boldsymbol{l} + (1 \leftrightarrow 2) \\ V_{\rm red}{}^{\rm si} &\approx (2m_1m_2)^{-1}\, p_i(X_{\rm v} - X_{\rm s})\, p_i - (4m_1m_2)^{-1}[p_i, [p_j, \delta_{ij}Y_{\rm v} + \hat{r}_i\hat{r}_j Z_{\rm v}]_+]_+ . \end{aligned} \tag{6.83}$$

Similar formulae for effective spin-dependent interactions, with a variety of starting assumptions, may be found in Schnitzer (1975, 1976a), Pumplin *et al* (1975), De Rújula *et al* (1975), Barbieri *et al* (1976) and Gromes (1977)†.

6.6. *Decay rate formulae*

In the computation of the decay rate from the approximate amplitude M a slight ambiguity enters with regard to relativistic kinematics, and the use of covariant against non-covariant phase space. To settle this, we recall that in a manifestly covariant treatment of a process $|I\rangle \to |F\rangle + \gamma$, with $|I\rangle$ and $|F\rangle$ single-particle boson states, the S-matrix element will have the form:

$$S_{FI} = -\mathrm{i}(2\pi)^4\, \delta(K_F + k - K_I)\ \mathscr{T} \tag{6.84}$$

where, with the usual kinematic factors for bosons extracted:

$$\mathscr{T} = (2\omega)^{-1/2}(2\mathscr{E}_F)^{-1/2}\ T(2\mathscr{E}_I)^{-1/2} \tag{6.85}$$

and T is the invariant Feynman amplitude. On comparison with our starting point (6.4), in which the bound-state wavefunctions are normalised via $\langle \psi_F | \psi_I \rangle = \delta_{FI}$, we are led to regard the amplitude eM as an approximation for the *invariant* quantity $T(4\mathscr{E}_F\mathscr{E}_I)_{\rm CM}{}^{-1/2}$, where the subscript indicates that the square root is evaluated in the rest frame of the decaying particle. In this frame the decay rate is given by:

$$R_{FI} = (2\pi)^4 \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \frac{\mathrm{d}\boldsymbol{K}_F}{(2\pi)^3}\, \delta(\boldsymbol{K}_F + \boldsymbol{k})\, \delta(m_I - \mathscr{E}_F - \omega)\, \Sigma' \frac{1}{2\omega} \frac{1}{2\mathscr{E}_F} |T|^2 \frac{1}{2m_I} \tag{6.86}$$

where Σ' denotes a sum over final spins and polarisations. With the identification:

$$T \approx (4\mathscr{E}_F(-\boldsymbol{k})\, m_I)^{1/2}\, eM \tag{6.87}$$

one gets:

$$R_{FI} = \frac{e^2}{\pi} \frac{k\mathscr{E}_F}{2M_I} \int \frac{\mathrm{d}\Omega}{4\pi}\, \Sigma' |M|^2 \tag{6.88}$$

where:

$$k = (m_I{}^2 + m_F{}^2)/2m_I \qquad \mathscr{E}_F = (m_I{}^2 + m_F{}^2)/2m_I . \tag{6.89}$$

Using the general form (6.80) for M and the easily derived relation:

$$\Sigma' |\langle \chi_F | \boldsymbol{\sigma}_1 . \hat{\boldsymbol{k}} \times \boldsymbol{\epsilon}^* | \chi_I \rangle|^2 = 2(2S_I + 1)^{-1} \tag{6.90}$$

† For a general analysis see Sucher (1978).

one finds:

$$R_{FI}=\frac{4\alpha Q^2}{2S_I+1}\frac{k^3}{m_c{}^2}\frac{\mathscr{E}_F}{m_I}\,|I_{\rm tot}|^2. \tag{6.91}$$

We should note that even if the quantity we have defined as 'M' had been computed exactly, the invariant T would not be given precisely by (6.87) because in our treatment we have neglected some dynamical recoil effects involving the final massive boson, e.g. by dropping the $\boldsymbol{K}$ dependence of the internal wavefunction $f_K(r)$ introduced in §6.1. However, one would expect such corrections to be of relative order $k^2/m_F{}^2$ or smaller. For application to the psions, the maximal value of k is $\sim 0{\cdot}75$ GeV and $m_F \gtrsim 2{\cdot}85$ GeV so that even in the worst case $\psi' \to \eta_c + \gamma$, $k^2/m_F{}^2$ is less than 7%. It will be a happy day for our understanding of the psions when corrections of this size become worthy of pursuit!

7. Application to the narrow resonances

7.1. Computation of M1 decay rates

We adopt the framework discussed in §5 in which the ψ and ψ' are described as the 1^3S_1 and 2^3S_1 bound states of a $c\bar{c}$ system, as well as the identification (5.18) and (5.17) of the $X(2{\cdot}83)$ with the 1^1S_0 state η_c and the $\chi(3{\cdot}45)$ with the 2^1S_0 state η_c', respectively, shaky as the last two assignments may be. The formalism developed in §6 can then be applied to the decays:

$$\begin{aligned} &\psi' \to \eta_c' + \gamma \qquad && \psi \to \eta_c + \gamma \\ &\psi' \to \eta_c + \gamma \qquad && \eta_c' \to \psi + \gamma. \end{aligned} \tag{7.1}$$

For simplicity we restrict ourselves to a V of the form (6.67) with Y_v and Z_v assumed not to exceed X_v in order of magnitude. Then *all* spin-dependent effects are of the order of v^4/c^4 and the formulae (6.80)–(6.82) for the equal-mass, opposite charge case may be used. The last two transitions in (7.1) are then 'relativistic M1 decays'. From the form of (6.73) and (6.82) one sees that *in order to make accurate predictions for these a great deal of information concerning the nature of the relativistic potential V is necessary*. Let us further put $X_s = 0$, so that:

$$V = X_v + Y_v \boldsymbol{\alpha}_1 . \boldsymbol{\alpha}_2 + Z_v \boldsymbol{\alpha}_1 . \boldsymbol{r} \boldsymbol{\alpha}_2 . \boldsymbol{r} \tag{7.2}$$

and the non-relativistic wavefunction $\tilde{\phi}(\boldsymbol{r})$ is determined by (6.76) with

$$U_0 = X_v(r). \tag{7.3}$$

Although (7.2) will give values for the $^3S_1$–$^1S_0$ splitting considerably smaller than the $\psi - X$ or $\psi' - \chi$ $(3{\cdot}45)$ mass difference our main purpose here is to illustrate, as simply as possible, the sensitivity of the M1 rates to the choice of V. Moreover, experimental values for the mass differences will be used where feasible. We shall consider two simple alternatives for (7.2):

(i) *V is a diagonal in Dirac matrices:*

$$Y_v = 0 \qquad Z_v = 0. \tag{7.4}$$

(ii) *V is a generalised Coulomb–Breit potential:*

$$Y_v = \tfrac{1}{3}(-2X_v - Z_v) \qquad Z_v = r^{-3}\int_0^r s^3 X_v'(s)\,ds. \tag{7.5}$$

While the first choice is an obvious one, the nomenclature and mysterious-looking relations in (7.5) require some discussion. Recall that the Breit interactions between electrons, arising from the exchange of a transverse photon, may be written in the form:

$$B=\frac{-(e^2/4\pi)}{2r}(\boldsymbol{\alpha}_1.\boldsymbol{\alpha}_2+\boldsymbol{\alpha}_1.\hat{\boldsymbol{r}}\boldsymbol{\alpha}_2.\hat{\boldsymbol{r}})=e^2\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\exp(\mathrm{i}\boldsymbol{k}.\boldsymbol{r})\,\alpha_1{}^i\alpha_2{}^jD_{ij}(\boldsymbol{k}) \tag{7.6}$$

where $D_{ij}(\boldsymbol{k})$ is the value at $k^0=0$ of the propagator $D_{ij}(k)=(\delta_{ij}-\hat{k}_i\hat{k}_j)/k^2$ for transverse photons, or equivalently:

$$B=-\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\exp(\mathrm{i}\boldsymbol{k}.\boldsymbol{r})(\boldsymbol{\alpha}_1.\boldsymbol{\alpha}_2-\boldsymbol{\alpha}_1.\hat{\boldsymbol{k}}\boldsymbol{\alpha}_2.\hat{\boldsymbol{k}})\,\tilde{V}_{\mathrm{c}}(\boldsymbol{k}) \tag{7.7}$$

where $\tilde{V}_{\mathrm{c}}(\boldsymbol{k})=e^2/\boldsymbol{k}^2$ is the Fourier transform of the Coulomb interaction:

$$\frac{e^2}{4\pi r}=\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\exp(\mathrm{i}\boldsymbol{k}.\boldsymbol{r})\,\tilde{V}_{\mathrm{c}}(\boldsymbol{k}). \tag{7.8}$$

Now suppose that in QCD, with the massless colour gluons described in an analogue of the Coulomb gauge, the static binding potential $U_0(r)$ (which at best has the form λ/r only at short distances) has a Fourier transform $\tilde{U}_0(\boldsymbol{k})$. A simple recipe (Schnitzer 1975) for generating spin-dependent interactions without introducing new parameters is to replace $\tilde{V}_{\mathrm{c}}(\boldsymbol{k})$ by $\tilde{U}_0(\boldsymbol{k})$ in (7.7). A straightforward calculation, with U_0 denoted by X_{v}, then yields a V of the form (7.2) with Y_{v} and Z_{v} given by (7.5). As a simple check, note that the choice $X_{\mathrm{v}}=\lambda/r$ in (7.5) gives $Y_{\mathrm{v}}=Z_{\mathrm{v}}=-\lambda/2r$ and the Breit operator is recovered.

The choices (i) and (ii) are complementary. The first yields $U_{\mathrm{ss}}=0$ so that $I_{\mathrm{spin}}=0$, but $I_{\mathrm{pair}}\neq0$:

$$\text{(i) } I_{\mathrm{spin}}=0 \qquad I_{\mathrm{pair}}=\langle\tilde{\phi}_F|-rX_{\mathrm{v}}'/6m_{\mathrm{c}}|\tilde{\phi}_I\rangle. \tag{7.9}$$

The second yields $U_{\mathrm{ss}}=\boldsymbol{\nabla}^2X_{\mathrm{v}}/6m_{\mathrm{c}}{}^2$ so that $I_{\mathrm{spin}}\neq0$, but now $I_{\mathrm{pair}}=0$:

$$\text{(ii) } I_{\mathrm{pair}}=0 \qquad I_{\mathrm{spin}}=\mp4\langle\tilde{\phi}_F|(X_{\mathrm{v}}''+2r^{-1}X_{\mathrm{v}}')/6m_{\mathrm{c}}{}^2|\tilde{\phi}_I\rangle(\tilde{W}_I-\tilde{W}_F)^{-1}. \tag{7.10}$$

The vanishing of I_{pair} in case (ii) is independent of X_{v} and corresponds to a generalisation of the cancellation found in the $2^3\mathrm{S}_1\rightarrow1^1\mathrm{S}_0+\gamma$ transition in He-like ions, between transverse photon and Coulomb terms arising from electron–electron interaction (§3.2.4).

7.1.1. Linear potentials. To get illustrative numerical results we take X_{v} to have the form (5.6) as in the early papers (e.g. Eichten *et al* 1975, Harrington *et al* 1975, Kogut and Susskind 1975, Kang and Schnitzer 1975a,b) but to simplify the presentation we drop the r^{-1} term:

$$X_{\mathrm{v}}=ar+b. \tag{7.11}$$

Then, as mentioned earlier, the Schrödinger equation (6.70) is solvable in terms of Bessel functions of fractional order and all integrals encountered may be done essentially exactly. If one introduces a dimensionless variable $\rho=r/\xi$ with the length ξ defined by:

$$\xi=(m_{\mathrm{c}}a)^{-1/3} \tag{7.12}$$

and writes:

$$\langle n'\ \mathrm{S}|r^p|n\ \mathrm{S}\rangle\equiv\xi^pC_{n'n}(p) \tag{7.13}$$

the constants $C_{n'n}^{(p)}$ are found to have the values (J S Kang 1977 private communication):

$$C_{11}^{(2)}=2{\cdot}916 \qquad C_{21}^{(2)}=-2{\cdot}560 \qquad C_{22}^{(2)}=8{\cdot}913 \tag{7.14}$$

$$C_{11}^{(1)}=1{\cdot}559 \qquad C_{21}^{(1)}=-0{\cdot}6532 \qquad C_{22}^{(1)}=2{\cdot}725 \tag{7.15}$$

and

$$C_{21}^{(-1)}=0{\cdot}3156. \tag{7.16}$$

These suffice for the evaluation of all terms, including $I_{\rm kin}$, since:

$$\langle 1S|\boldsymbol{p}^2/m_c|2S\rangle=\langle 1S|-X_v|2S\rangle=-a\langle 1S|r|2S\rangle \tag{7.17}$$

and, as follows from the virial theorem:

$$\langle nS|\boldsymbol{p}^2/m_c|nS\rangle=\tfrac{1}{2}\langle nS|rX_v'|nS\rangle=\tfrac{1}{2}a\langle nS|r|nS\rangle. \tag{7.18}$$

So far, the values of the parameters m_c, a and b have not been specified. With $r=\xi\rho$, the S-wave equation for $u(\rho)=\tilde{\phi}(r)/r$ obtained from (6.70) with $U_0=ar+b$, reads $u''(\rho)+\rho u=\epsilon u$ with $\epsilon=m_c\xi^2(\tilde{W}-b)$. The lowest eigenvalues are $\epsilon_1=2{\cdot}238$, $\epsilon_2=4{\cdot}088$, so that on equating $m_{\psi'}-m_\psi\approx 586$ MeV with $\tilde{W}_2-\tilde{W}_1$ one gets $m_c\xi^2=2{\cdot}99$ GeV^{-1}. The leptonic decay rate for $\psi\to l^+l^-$ is usually approximated by the positronium-like formula:

$$\Gamma(^3S_1\to l^+l^-)=3(16\pi\alpha^2/3m_I^2)\,Q^2|\tilde{\phi}(\boldsymbol{r}=0)|^2 \tag{7.19}$$

where the factor of 3 arises from colour degrees of freedom. For a linear potential, the S-state identity $|\tilde{\phi}(\boldsymbol{r}=0)|^2=(\mu/2\pi)\langle\tilde{\phi}|U_0'|\tilde{\phi}\rangle$ ($\mu=\frac{1}{2}m_c$ is the reduced mass) yields $|\tilde{\phi}(\boldsymbol{r}=0)|^2=am_c/4\pi$. Thus $\Gamma(^3S_1\to l^+l^-)=4\alpha^2Q^2\xi^{-3}/m_I^2\simeq 4{\cdot}8$ keV implies, with $Q=\frac{2}{3}$, that $\xi=1{\cdot}27$ GeV^{-1} and hence:

$$m_c=1{\cdot}85\text{ GeV} \qquad a=0{\cdot}262\text{ GeV}^2. \tag{7.20}$$

A fit to the mass m_ψ via $m_\psi=2m_c+\tilde{W}_1+b$ then gives:

$$b=-1{\cdot}38\text{ GeV}. \tag{7.21}$$

The resulting values for the M1 decay rates are shown in the last column of table 2, which also displays the values of the various terms contributing to $I_{\rm tot}$, defined by the sum shown in (6.81). In the evaluation of the I and the rate we have used the experimental mass values 3684 ± 5, 3454 ± 10, 3098 ± 3 and 2830 ± 30 MeV, for ψ', η_c', ψ and η_c respectively and computed k from (6.89) in order to get somewhat more realistic results, since the observed mass splittings are considerably larger than would be given by a consistent use of (7.2) and (7.11). In this connection, we should note that for $\psi'\to\eta_c+\gamma$ the entry under $I_{\rm ret}$ in table 2 denotes the value of

$$\langle 1S|1-j_0(kr/2)|2S\rangle$$

rather than the leading term $\langle 1S|k^2r^2/24|2S\rangle$. Because $\langle(kr/2)^2\rangle\sim 0{\cdot}6$ an expansion in powers of k^2 is only slowly convergent. Moreover, use of the leading term alone yields an almost exact cancellation between $I_{\rm pair}$, $I_{\rm kin}$ and $I_{\rm ret}$ and hence to a rate which seems improbably small.

Inspection of table 2 shows that for the unhindered decays, $\psi'\to\eta_c'+\gamma$, $\psi\to\eta_c+\gamma$, where $I_{\rm pair}$, $I_{\rm kin}$ and $I_{\rm ret}$ represent $\mathcal{O}(v^2/c^2)$ corrections to the leading term '-1', there is not much difference between cases (i) and (ii), the effect being a reduction of 10–20% of the zero-order rate. However, for the hindered decay $\eta_c'\to\psi+\gamma$, the computed rates differ by a factor of three, while for $\psi'\to\eta_c+\gamma$ they differ by a factor

Table 2. M1 decay rates of psions obtained from a relativistic interaction V which is (i) diagonal in Dirac matrices or (ii) a generalised Coulomb–Breit potential, defined by (7.5). The quantity I_{tot} is the sum of the first four entries, as in (6.81) of the text and the rate is computed from (6.91) of the text.

Decay		I_{dir}'	I_{pair}	I_{kin}	I_{ret}	I_{tot}	Rate (keV)
$\psi' \to \eta_c' + \gamma$	(i)	−1·00	−0·082	0·164	0·030	−0·89	10·3
	(ii)	−1·00	0	0·164	0·030	−0·81	8·5
$\psi \to \eta_c + \gamma$	(i)	−1·00	−0·047	0·094	0·013	−0·94	17·2
	(ii)	−1·00	0	0·094	0·013	−0·89	15·4
$\eta_c' \to \psi + \gamma$	(i)	0	0·020	0·079	−0·020	0·079	0·8
	(ii)	0·071	0	0·079	−0·020	0·130	2·1
$\psi' \to \eta_c + \gamma$	(i)	0	0·020	0·079	−0·086†	0·013	0·07
	(ii)	−0·030	0	0·079	−0·086†	−0·37	0·60

† This entry includes some higher-order retardation effects as explained in the text.

of *ten*. This illustrates the sensitivity of these decays to the choice of the relativistic potential V.

Another simple example of the sensitivity of hindered M1 decays to the form of V is provided by (iii) *a relativistic scalar potential* $V = \beta_1\beta_2 X_s(r)$. This gives $U_0 = X_s(r)$, $U_{ss}(r) = 0$ and, with $X_s(r) = ar + b$, the values of I_{dir}', I_{kin} and I_{ret} are as for case (i), while:

$$I_{\text{pair}} = \langle n'\text{S} \,|\, X_s/m_c \,|\, n\text{S}\rangle = (a\xi C_{n'n}^{(1)}/m_c) + (b/m_c)\,\delta_{n'n}. \tag{7.22}$$

Computation yields $R(\psi' \to \eta_c' + \gamma) \approx 15$ keV, $R(\psi \to \eta_c + \gamma) \approx 36$ keV and $R(\eta_c' \to \psi + \gamma) \sim 0{\cdot}4$ keV, all within a factor of two of the rates for case (i), but $R(\psi' \to \eta_c + \gamma) \approx 6$ keV which is larger than the value found for case (i) by a factor of 100†.

In this connection it should be noted that the additive constant 'b', which is needed to get a fit to the total mass of the ψ and presumably compensates for dynamical effects not included in the ar term, enters explicitly into the formulae for the decay amplitudes only for the scalar case (iii) and $\psi' \to \eta_c' + \gamma$, $\psi \to \eta_c + \gamma$, via (7.20). Since the origin of 'b' is obscure, the numbers for these cases should be viewed with extra caution.

We next consider some physical implications of these results.

7.2. Comparison with experiment

The experimental situation with regard to the M1 decays is as follows. On the one hand, there is positive evidence for the transition $\psi \to X\gamma$, indeed for the very existence of the $X(2{\cdot}83)$, which has been assigned to η_c in (7.1), coming from the observation of peaks in the energy distribution of three-photon final states which can be interpreted as arising from the sequence $\psi \to X\gamma$, $X \to \gamma\gamma$. Such experiments (Braunschweig *et al* 1977) provide only a value for the product of branching ratios:

$$\text{BR}(\psi \to X\gamma)\ \text{BR}(X \to \gamma\gamma) = 1{\cdot}2 \pm 0{\cdot}5 \times 10^{-4}. \tag{7.23}$$

† The numbers quoted here and in table 2 are updated and, in a few cases, corrected values for results given in Feinberg and Sucher (1975a) and Sucher (1976).

There is similar evidence (Whitaker *et al* 1976) for the existence of the $\hat{\chi} \equiv \chi(3{\cdot}45)$, assigned to η_c in (7.1) and the transitions $\psi' \to \hat{\chi}\gamma$, $\hat{\chi} \to \psi\gamma$, with a branching ratio product given by:

$$\text{BR}(\psi \to \hat{\chi}\gamma)\,\text{BR}(\hat{\chi} \to \psi\gamma) = 0{\cdot}8 \pm 0{\cdot}4 \times 10^{-2}. \tag{7.24}$$

On the other hand, from study of the inclusive photon spectrum the following upper bounds have been obtained (Biddick *et al* 1977):

$$\text{BR}(\psi' \to \hat{\chi}\gamma) < 2{\cdot}5\% \tag{7.25}$$

$$\text{BR}(\psi' \to X\gamma) < 1\% \tag{7.26}$$

$$\text{BR}(\psi \to X\gamma) < 1{\cdot}7\%. \tag{7.27}$$

Since $\Gamma(\psi' \to \text{all}) = 228 \pm 56$ keV and $\Gamma(\psi \to \text{all}) = 67 \pm 12$ keV are both known it is easy to compare these bounds with theoretical expectations from the simple models we have considered. From (7.25) one infers that $\Gamma(\psi' \to \hat{\chi}\gamma) < 5{\cdot}7 \pm 1{\cdot}4$ keV which is roughly compatible with the rates shown in table 2, about 8–10 keV. Similarly from (7.26) one gets $\Gamma(\psi' \to X\gamma) < 2{\cdot}3 \pm 0{\cdot}6$ keV which is easily compatible with the rates 0·1–0·6 keV shown in table 2. However, for the ordinary M1 decay $\psi \to X\gamma$, (7.27) implies $\Gamma(\psi \to X\gamma) < 1{\cdot}1 \pm 0{\cdot}2$ keV which is smaller than either entry in table 2 by more than a factor of ten.

The other experimental numbers represent difficulties not so much for a composite model of the psion *per se*, but rather for the estimate of hadronic decay widths within the QCD framework. From (7.23) and (7.27) one gets a lower bound $\text{BR}(X \to \gamma\gamma) > (0{\cdot}7 \pm 0{\cdot}3)\%$ whereas a QCD estimate (Appelquist and Politzer 1975a,b) is $\text{BR}(X \to \gamma\gamma) \sim 0{\cdot}13\%$, which is a factor of three to eight smaller than the lower bound. Notice that this is *independent* of any estimate for radiative decay and would remain a difficulty even if the theoretical value of $\Gamma(\psi \to X\gamma)$ were brought down.

If we further use table 2 to estimate $\Gamma(\psi' \to \eta_c'\gamma) \sim 10$ keV and so to infer $\text{BR}(\psi' \to \eta_c'\gamma) \sim 4 \pm 2\%$ we get, using (7.24), $\text{BR}(\hat{\chi} \to \psi\gamma) \sim 20\%$. This implies, on use of the estimate $\Gamma(\hat{\chi} \to \psi\gamma) \sim 1$ keV, that $\Gamma(\hat{\chi} \to \text{hadrons}) \sim 5$ keV—a few hundred times smaller than the QCD estimate of 2·4 MeV.

Our discrepancy factors are not as large as those of, for example, Gottfried (1977), who keeps the r^{-1} term in (5.6), which we have dropped to simplify the presentation, but uses M1 decay rates which include only the retardation effect. As can be inferred from table 2, for the case of *no* r^{-1} term, the inclusion of only I_{ret} for $\eta_c' \to \psi\gamma$ would yield a decay rate of about 0·05 keV (roughly the same as the 0·1 keV cited by Gottfried) rather than the ~ 1 keV value shown. Thus for this particular hindered transition inclusion of the other effects discussed *increases* the rate by an order of magnitude; further improvement occurs if the r^{-1} term is kept in the analysis (J S Kang 1977 private communication, Kang and Sucher 1978). However, the overall conclusion (Chanowitz and Gilman 1976, Jackson 1977, Gottfried 1977) is not changed. Assignment of the X and $\chi(3{\cdot}45)$ to the η_c and η_c' poses a problem for the $\psi \to \eta_c\gamma$ rate and a severe difficulty for the QCD estimate of $\eta_c' \to$ hadrons.

7.3. Survey of present status

As we have seen, the pioneer model described in §5.1 has turned out to be inadequate from a quantitative point of view. There are difficulties with (*a*) the spectrum of states, (*b*) the width for some radiative decays, and also with (*c*) the QCD picture for

some hadronic decays, involving $c\bar{c}$ annihilation into virtual colour gluons followed by recombination into the final hadron state. We shall only mention a few of the theoretical efforts and suggestions that have been made to deal with these issues. First, the E1 widths have been shown to be strongly affected by the coupling to other channels, above the threshold for production of charmed particles, which of course is not taken into account by the simple potential model; inclusion of this coupling reduces the widths considerably, although $\Gamma_{\text{th}}(\psi' \to \chi(3{\cdot}41)+\gamma)$ still exceeds the experimental width (~ 17 keV) by a factor of two or so (Eichten *et al* 1976, 1977). Next, it has been suggested that the quarks have a large anomalous 'colour moment' ($\kappa \sim 1$) as far as their coupling to gluons is concerned; then the large $\psi - \eta_c$ mass difference can be accounted for (Schnitzer 1976b). While the ratio of the P-state splittings is also improved, their magnitude becomes too large (Pham and Richard 1977). Another simple possibility for getting a large spin–spin interaction is the exchange of *axial vector* gluons by the quarks (Feinberg and Sucher 1975a,b). (Axial colour gluons arise naturally within unified gauge theories which utilise 'chiral' colour gauges (Pati and Salam 1975).) A very recent suggestion is that a large spin–spin force can arise from the exchange of 'instantons' (Wilczek and Zee 1978). The idea that the non-relativistic potential gets significant contributions from a relativistic *scalar* interaction has also been further explored (Henriques *et al* 1976) and a fit to the spectrum using both scalar exchange *and* a large value of 'κ' has been obtained (Chan 1977, Carlson and Gross 1978).

No definitive picture has yet emerged regarding these issues. For comprehensive reviews with extensive references see Jackson (1976a, 1977) and Gottfried (1977). We mention only some recent work which may be of particular interest to atomic physicists also: the derivation of new theorems on the ordering of levels for the non-relativistic Schrödinger equation for a wide class of potentials (Martin 1977, Grosse 1977) and the application of ancient sum rule and dispersion theory techniques, first used in atomic physics, to obtain results of interest for the E1 radiative decays (Jackson 1976b) and e^+e^- annihilation (Novikov *et al* 1977a,b, Farrar *et al* 1977).

Of course, further investigation of the connection between QCD and the phenomenological binding potential (Duncan 1976, Feinberg 1977, Appelquist *et al* 1977, Fischler 1977, Poggio 1977) is of considerable general interest.

8. Concluding remarks

Relativistic magnetic dipole transitions in H-like and He-like ions offer an interesting field of application for quantum electrodynamics. The tools needed for a calculation of the decay rates in leading order have been available since the early 1930s. However, it appears that only in the present decade, under the stimulus/constraint of *terrestrial experiments*, was the correct value of the rate for even as simple a process as $2^1S_{1/2} \to 1^1S_{1/2}+\gamma$ in hydrogen worked out. The reader may construct her or his own moral from this fact. At present the theory for the lifetime of the 2^3S_1 states, as described in §3, is in excellent agreement with beam–foil experimental values, for He-like ions, which now cover three orders of magnitude ($Z=16$ to $Z=36$). When the measured decay rate for helium itself is included, the agreement extends to fourteen orders of magnitude. The theory may therefore be used with confidence for the purpose of calculating rates of interest for astrophysics or solar physics. With regard to the latter, it is of interest to note that the initial evidence for $2^3S_1 \to 1^1S_0+\gamma$ in

He-like ions was in the region $Z=5$ to $Z=13$, which falls in between $Z=2$ for He and $Z=16$ for S, the lowest value measured in beam–foil experiments. However, there is now also evidence for this transition in He-like Fe ions ($Z=26$) in the solar corona (Bhalla *et al* 1975) so that one has at least one example of an Earth–Sun overlap.

Although the relativistic M1 transitions $2^3S_1 \rightarrow 1^1S_0 + \gamma$ in the helium isoelectronic sequence may now be regarded as well understood, there are other areas in atomic physics in which similar transitions play a role and which merit further investigation. For example, although as mentioned before, the relativistic $2^1S_{1/2} \rightarrow 1^1S_{1/2} + \gamma$ transition in a hydrogenic ion has not yet been seen because the competition with two-photon decay requires that Z be extremely large, an analogous inner-shell x-ray transition can be observed in heavy atoms, as reported by Boehm (1970) for Tm ($Z=69$). The theoretical calculation of the decay rate (Rosner and Bhalla 1970) is then however beset with the usual difficulties encountered in many-electron atoms, which require use of Hartree–Fock or related approximation schemes. An interesting exception to this is provided by the case of a μ–mesic atom, with the muon in a low Bohr orbit, so that interaction with the electron cloud is minimal and relatively reliable estimates can be made. The $2^1S_{1/2} \rightarrow 1^1S_{1/2} + \gamma$ decay in such an atom is of particular interest because of the possibility of seeing the effects of a parity-violating μ–nucleon interaction by, for example, detection of a circular polarisation for the emitted photon, which could arise from mixing with the 2P state (Feinberg and Chen 1974). The search for parity-violating effects in ordinary atoms, a subject pioneered by Bouchiat and Bouchiat (1974, 1975), is currently an intense area of investigation. This is because in the pioneer unified gauge theory of weak and electromagnetic interactions (Weinberg 1967, Salam 1968), which has proved to be very successful in correlating high-energy neutrino–nucleon interactions, one expects the existence of measurable parity-violating effects in such atoms, arising from neutral currents. The characteristics of relativistic M1 transitions enter into the analysis of some of the ongoing experiments, involving the circular dichroism of atoms with large Z. The interpretation of these is unfortunately again complicated by the need to include many-electron effects. For recent reviews of these matters see G Feinberg (1977) and Sandars (1977).

The definitive state of affairs for M1 transitions in He-like ions may be contrasted with the present understanding of the radiative transitions between the narrow resonances in the 3–4 GeV range, as described in §§5 and 7. The study of the hydrogen atom (and positronium) gave great confidence in the validity of QED as an extremely precise description of the interactions of importance in atomic physics. In the early days it was hoped that the ψ would prove to be the long sought 'hydrogen atom of hadronic physics', i.e. a system whose properties could be calculated with considerable accuracy from an underlying fundamental theory, such as QCD, and thereby do the same job for QCD that the H atom did for QED. This hope, although still very much alive, has so far not been realised. It was based in part on the idea that because of the presumably large mass of the c, the motion of the c and $\bar{c}$ quarks could be treated, in a good first approximation, by a simple non-relativistic dynamics.

Pending derivation of the 'true potential' from a fundamental theory, it would be useful to search for a phenomenological potential with only a small number of parameters which would give *both* spectrum and radiative decay rates in agreement with experiment. In such a search the constraints provided by the need to fit the M1 decays of the psions will play an important role. The theory of such decays has been presented in §6 in a form which largely avoids commitment to a specific choice of relativistic

potential, with the hope that it will prove useful in this context as well as in the analysis of radiative decays of other physical systems which may admit a description as two-body bound states.

In this connection it should be mentioned that in the summer of 1977 a new narrow resonance—called upsilon (Υ)—was observed in a reaction analogous to that involving the discovery of the J/ψ:

$$\text{p} + \text{nucleus} \rightarrow \Upsilon + \text{hadrons} \qquad \Upsilon \rightarrow \mu^+\mu^-$$

as a peak in the $\mu^+\mu^-$ spectrum at 9·6 GeV (Lederman 1977, Herb *et al* 1977). There is evidence for at least one and possibly two further peaks at around 10 GeV (Innis *et al* 1977). The Υ and its partners may be interpretable as $(b, \bar{b})$ bound states where 'b' denotes a truly heavy quark, with $m_b \sim 5$ GeV. In the study of the dynamics of such a system M1 radiative decays may again play a significant role, because of their sensitivity to structure and hence to assumptions about the binding interactions. Will this system prove to be the H atom of hadronic physics?

Whatever the answer, it seems safe to conclude that, in one way or another, M1 transitions of the kind which have been studied in this review will continue to be of interest for some time to come, in atomic physics as well as in elementary particle physics and especially at the interface of these two fields.

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