Modeling the Dynamics and Gravitational-Wave Emission of Coalescing Binary Black Holes

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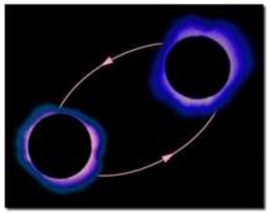
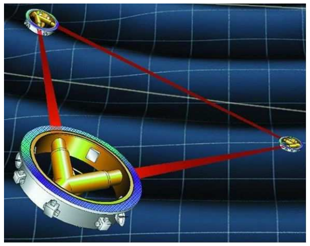
Contents

- **The *problem of motion*, i.e., the problem of describing the dynamics of gravitationally interacting bodies, is the cardinal problem of any theory of gravity. In general relativity, the *gravitational-wave generation problem* also plays a fundamental role.**
- **Analytical modeling of the two-body problem: from Kepler to Newton to Einstein.**
- **Have the two-body problem and the generation problem been *solved exactly* in general relativity? Which analytical approximation tools exist to tackle those problems?**
- **Is it possible to describe analytically the inspiral, plunge, merger, and ringdown of *comparable mass* black holes?**



- **Why is it relevant in astrophysics and in the search for gravitational waves to solve those problems as accurately as possible?**

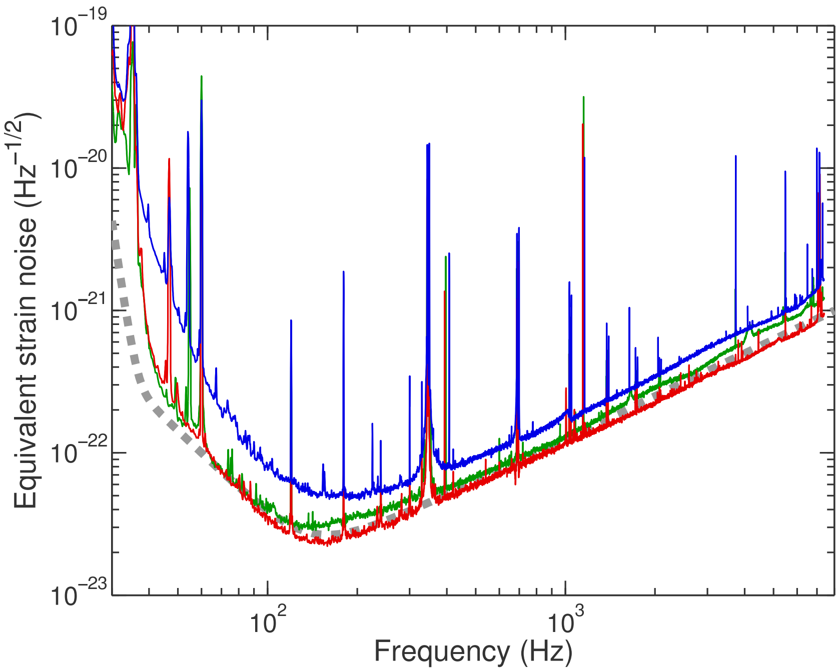


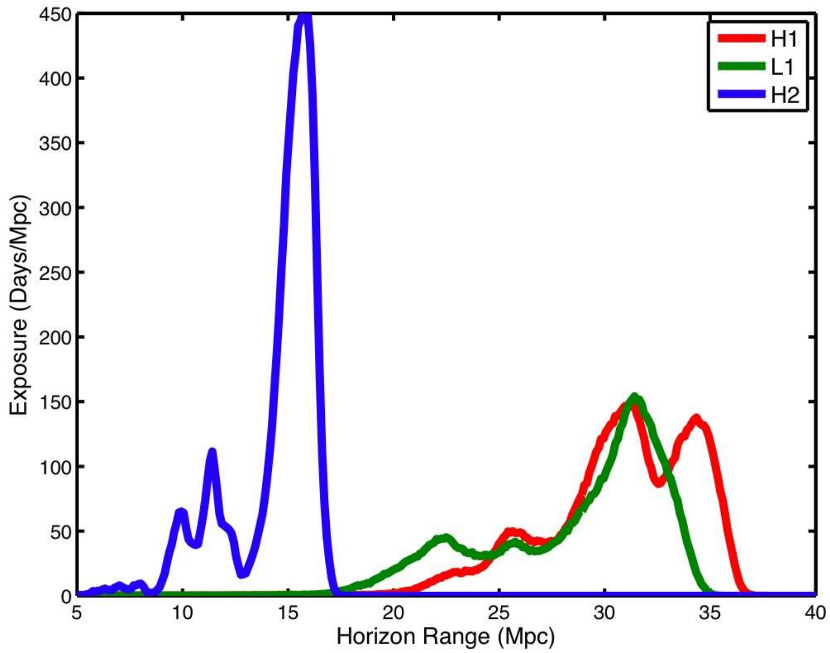
International network of GW interferometers (frequency band ~ 10–10^3 Hz)

LIGO at Livingston (LO) ⇒

⇐ LIGO at Hanford (WA)

LIGO detectors during the most sensitive run (S5)





Astrophysical reach of the S5 run: [LIGO Scientific Collaboration 08]

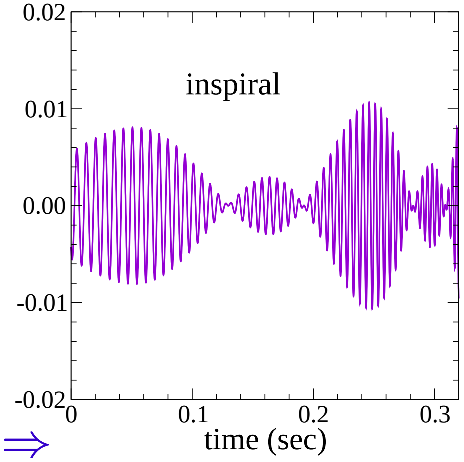
NS-NS ($1.4 M_\odot + 1.4 M_\odot$): 31 Mpc (~ 147 milky-way–equivalent galaxies)

BH-BH ($10 M_\odot + 10 M_\odot$): 125 Mpc (~ 12,000 milky-way–equivalent galaxies)

First inspiral-merger-ringdown search under completion. Results coming out soon!

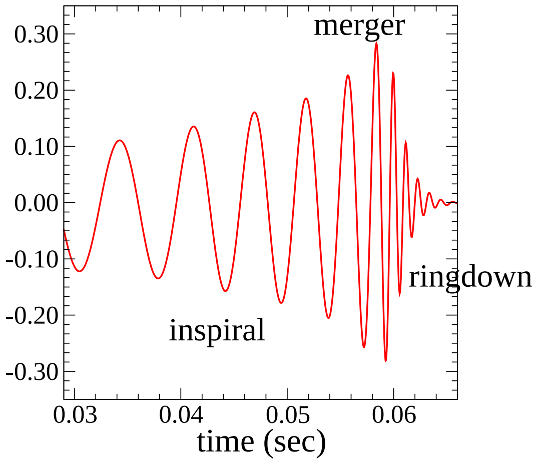
Why is it crucial to know in advance the gravitational waveforms of coalescing binaries?

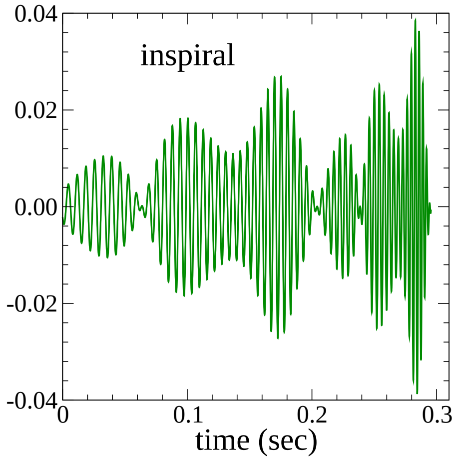
- **The detectors' noise level prevents observing the waveforms *directly*. Match-filtering techniques are used to dig out the waveforms from the noise.**



- **Different shapes, different binary parameters** $(m_1, m_2, \mathbf{S}_1, \mathbf{S}_2) \Rightarrow$

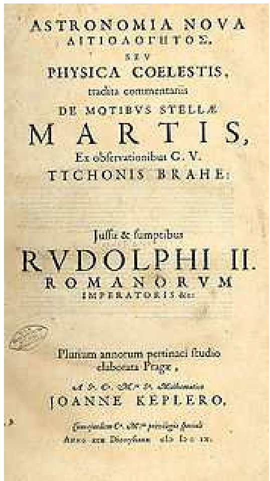
$\Downarrow$





- **GWs can tell us how *heavy* each of the black holes was; how *fast* they were spinning; the *shape* of their *orbit* (circular? elongated?); *where* the holes were in the sky, and *how far* they were from Earth.**

Milestones in understanding the problem of motion



ASTRONOMIA NOVA
ΑΙΤΙΟΛΟΓΗΤΟΣ,
SEV
PHYSICA COELESTIS,
tradita commentariis
DE MOTIBVS STELLÆ
MARTIS,
Ex obſervationibus G. V.
TICHONIS BRAHE:

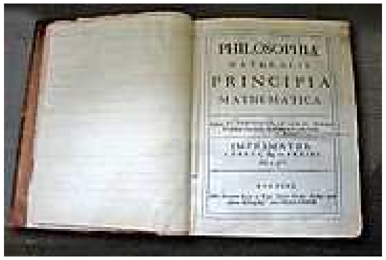
Juſſu & ſumptibus
RVDOLPHI II.
ROMANORVM
IMPERATORIS &c:

Plurium annorum pertinaci ſtudio
elaborata Pragæ,

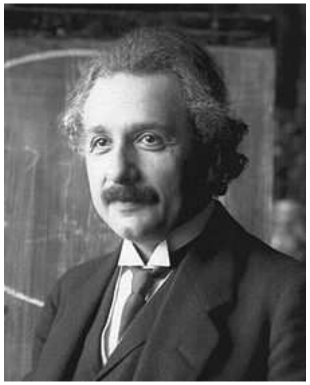
JOANNE KEPLERO,

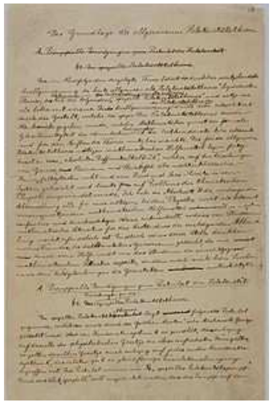
Laws of planetary motion

Gravitation



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA





General relativity

The problem of motion in Newtonian gravity

- *Two-body* **Hamiltonian:** $H_{\rm Newt} = \frac{1}{2m_1}\,\mathbf{p}_1^2 + \frac{1}{2m_2}\,\mathbf{p}_2^2 + U(r)\,, \qquad U(r) = -\frac{G m_1\, m_2}{r}$
- **Reduction to** *one-body* **Hamiltonian:**

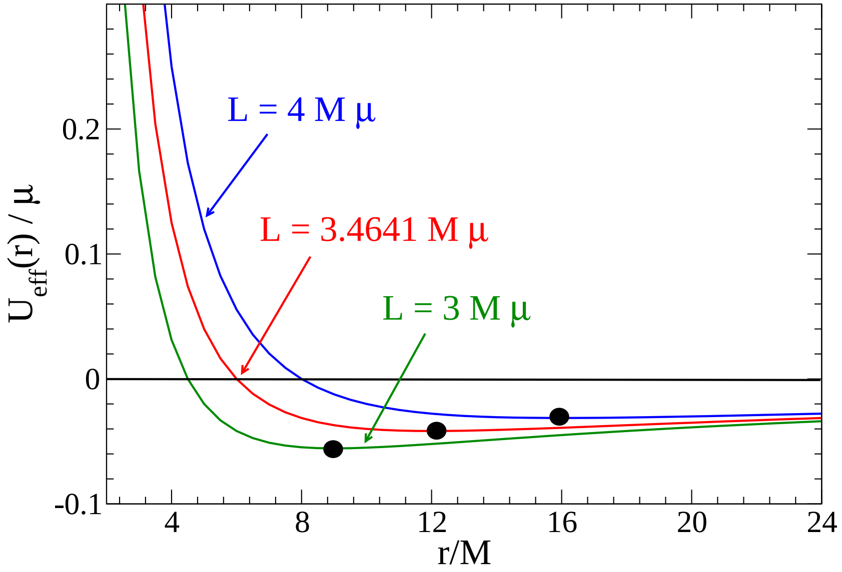
$$\mathbf{r}_{\rm CM} = \frac{m_1\,\mathbf{r}_1 + m_2\,\mathbf{r}_2}{M}, \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \qquad M = m_1 + m_1, \qquad \mu = m_1\,m_2/M$$

$$H_{\rm Newt} = \frac{1}{2\mu}\,\mathbf{p}^2 + U(r)$$

- $H_{\rm Newt}(\mathbf{r},\mathbf{p})$ **describes a test-particle of mass** μ **orbiting an** *external* **mass** M
- **Effective radial potential:**

$$\frac{U_{\rm eff}(r)}{\mu} = \frac{1}{2}\,\frac{L^2}{\mu^2 r^2} - \frac{M}{r}$$

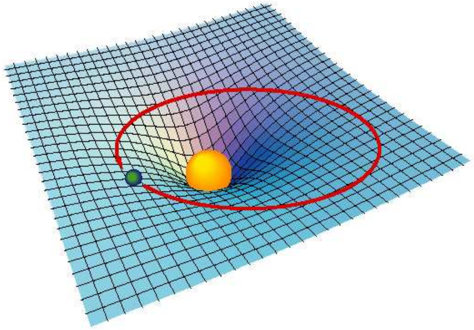
- **Bounded orbits are closed**
- **For any angular momentum** $L \neq 0$ **there exists a circular orbit**

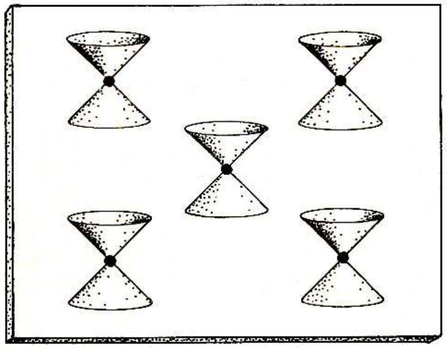


General Relativity

- **1907-1915: Einstein develops the theory of general relativity**

  Spacetime is no longer given *a priori* **and it is no longer independent of all material content. Spacetime is a dynamic entity both influencing and influenced by the distribution of mass-energy that it contains.**



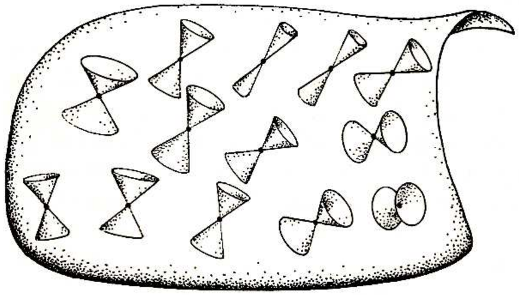


$$\Leftarrow ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$\eta_{\mu\nu} = (-1, +1, +1, +1)$$

$$\mu, \nu = 0, 1, 2, 3$$

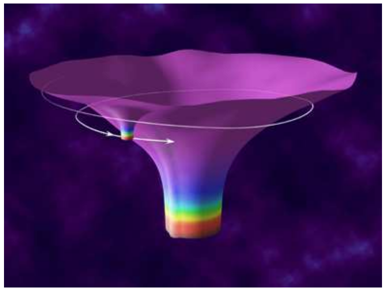
$$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu \quad \Rightarrow$$



Rigid spacetime geometry in special relativity

Elastic spacetime geometry in general relativity

The problem of motion in Einstein gravity



- **Schwarzschild metric (solution of Einstein equations)**

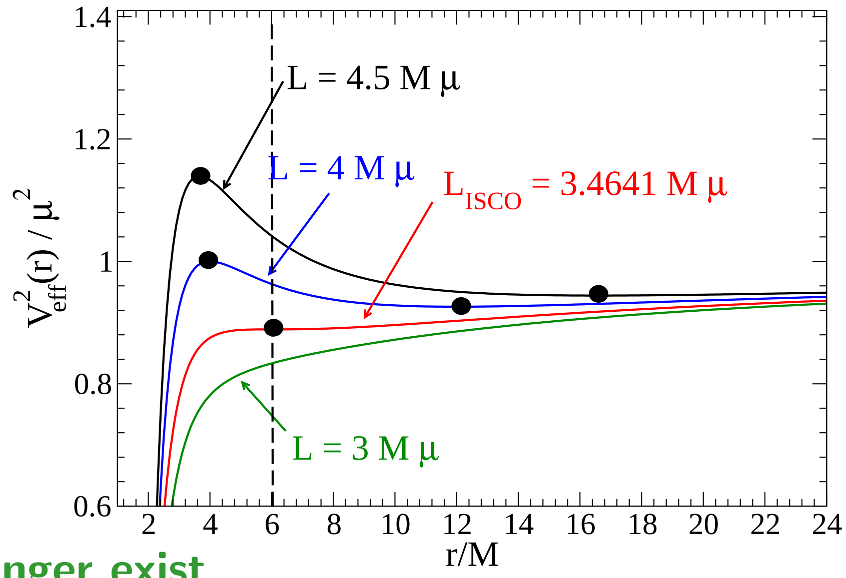
$$ds^2 = -\left(1-\frac{2GM}{rc^2}\right)\,c^2\,dt^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1}\,dr^2 + r^2\,d\Omega^2$$

$$H_{\rm Schw}(\mathbf{r},\mathbf{p}) = \mu\,\sqrt{\left(1-\frac{2M}{r}\right)\,\left[1+\frac{\mathbf{p}^2}{\mu^2}-\frac{2M}{r}\,\frac{p_r^2}{\mu^2}\right]}$$

- **$H_{\rm Schw}(\mathbf{r},\mathbf{p})$ describes a test-particle of mass μ orbiting a black hole of mass M**

- **Effective radial potential:**

$$\frac{V^2_{\rm eff}(r)}{\mu^2} = \left(1-\frac{2M}{r}\right)\,\left(1+\frac{L^2}{\mu^2 r^2}\right)$$



- **Bounded orbits are *not* closed**
- **There exists an angular momentum $L_{\rm ISCO}$ such that for $L < L_{\rm ISCO}$ circular orbits no longer exist**

Gravitational waves: prediction of general relativity

- **In 1916 Einstein predicts the existence of gravitational waves:** *elastic* **spacetime**

Linearized gravity: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \quad \Rightarrow \quad \Box \tilde{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

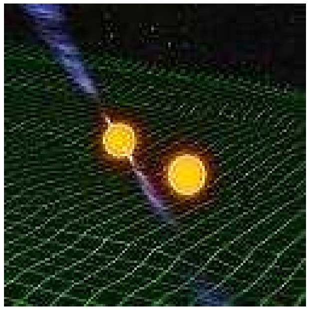
- **Analogy with Maxwell equations:** $\Box A_\mu = -4\pi J_\mu$

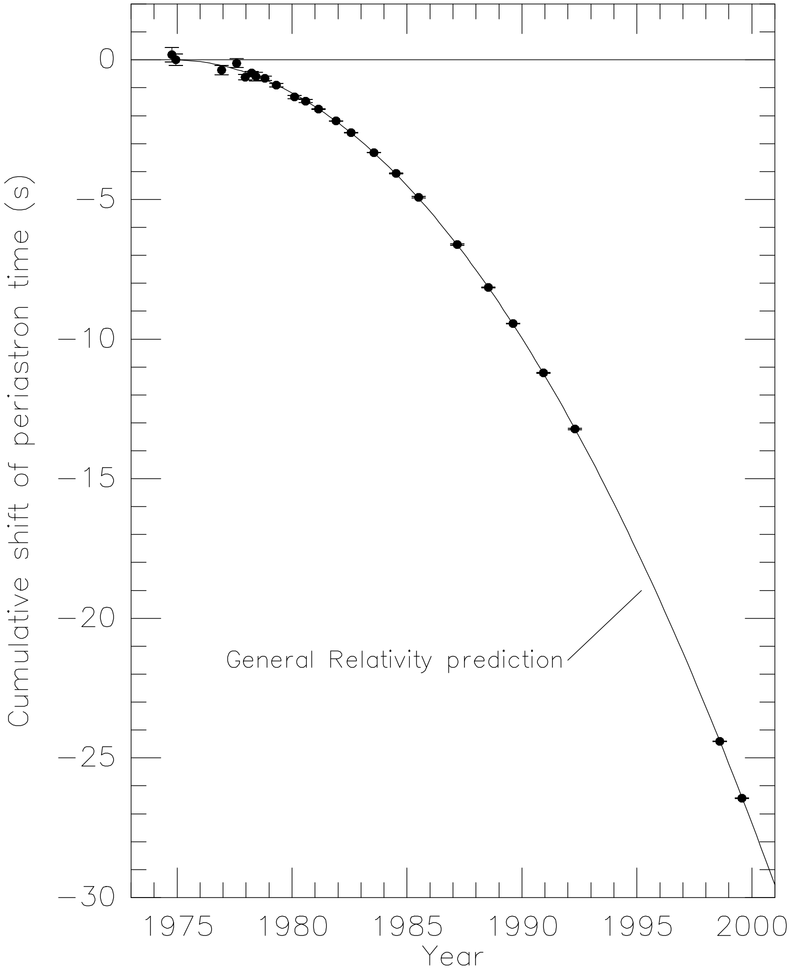
- **A distribution of mass deforms the spacetime geometry in its neighborhood. Also, this deformation propagates at finite speed out to infinity in the form of waves whose oscillations reflect the temporal variation of the matter distribution.**

⇐ **Ripples of space-time curvature from two black holes orbiting each other**

Ripples of water from a stone thrown in the water ⇒

The binary pulsar PSR1913+16



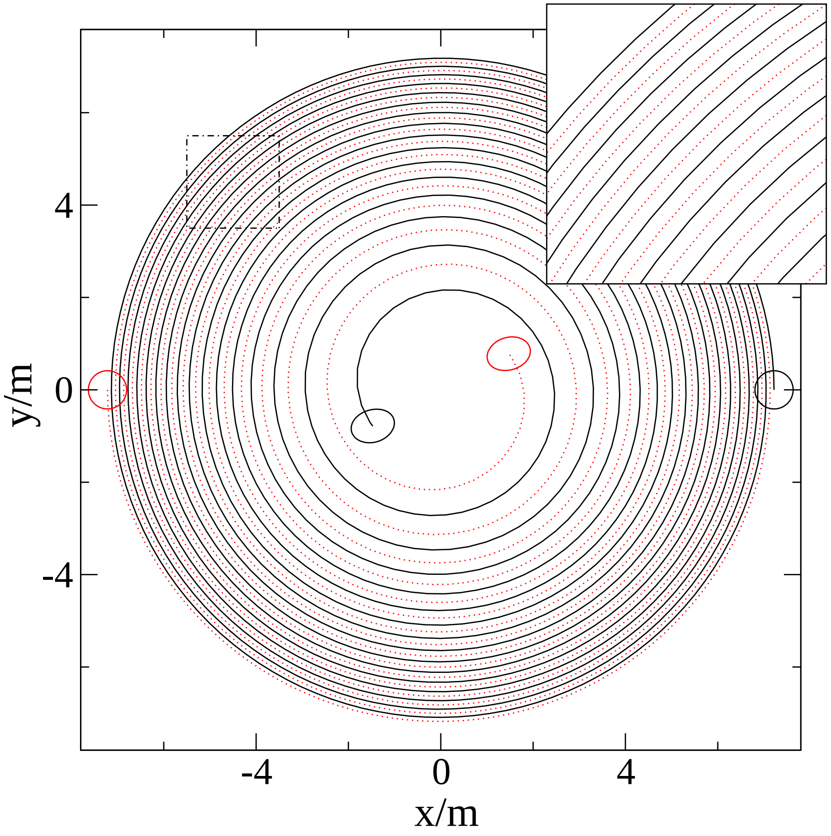


- **A radio pulsar in a close orbit with period 8 hours around an unseen companion**
- **Discovered in 1974 by Hulse and Taylor**
- **Long-term radio observations have yielded objects' masses and orbital parameters**
- **The orbital period is slowly decreasing at just the rate predicted by general relativity**
- **The strongest evidence for the existence of gravitational radiation**

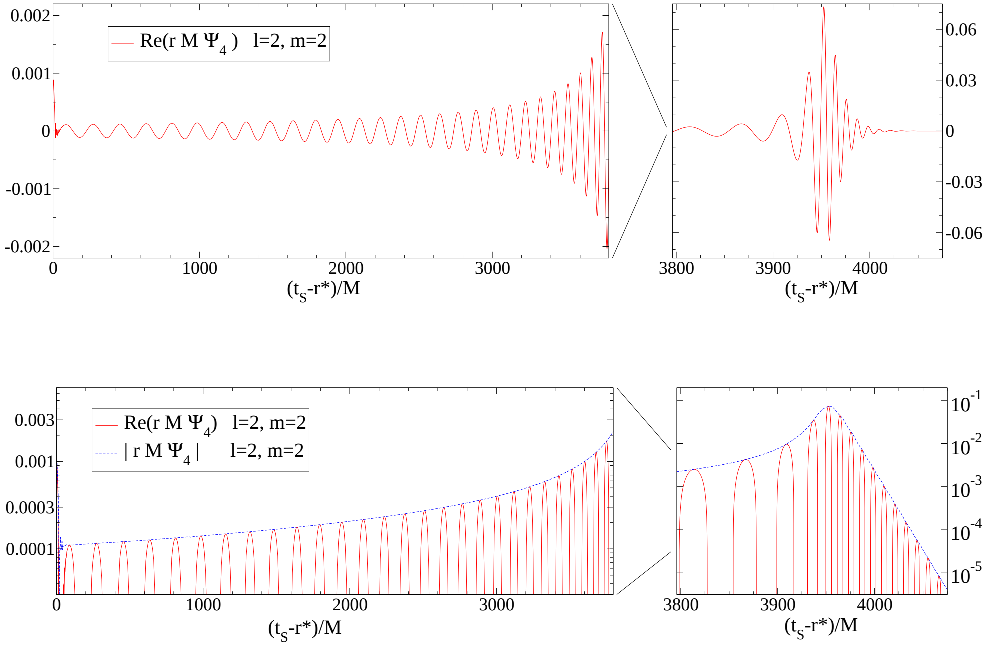
The motion and generation problems: solving Einstein equations *numerically*

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- **Breakthrough in numerical relativity in 2005-2006**



Inspiral-merger-ringdown waveform



`[Caltech-Cornell collaboration 08]`

Post-Newtonian approximation

- **First developed by Einstein, Droste and De Sitter in 1917**
- **Newtonian gravity is recovered assuming a weak deformation of flat spacetime, a slowly varying field, and a non-relativistic source:**

$$\Delta h_{00} \simeq -\frac{8\pi G}{c^4} T_{00} \quad \Rightarrow \quad \Delta U = -4\pi\rho$$

- **PN expansion: working within the Newtonian limit and keeping terms of higher order in the parameter**

$$\epsilon \sim \frac{v^2}{c^2} \sim \frac{GM}{c^2 r} \sim \frac{(\text{time derivative of h})^2}{(\text{space derivative of h})^2} \sim \frac{(\text{momentum})^2}{(\text{energy})^2}$$

- **PN force of gravitational attraction between two bodies:**

$$F_{\rm PN} = -\frac{G\, m_1\, m_2}{r^2}\left(1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^5}{c^5} + \frac{v^6}{c^6} + \frac{v^7}{c^7} + \cdots\right)$$

v^5/c^5 **(2.5 PN order) ⇒ radiation-reaction effects**

- **1PN prediction tested in solar system ($\frac{v}{c} \sim 10^{-5}$–10^{-4}) and binary pulsars ($\frac{v}{c} \sim 10^{-3}$)**

How far in the strong-field regime can we push the PN approximation?

- **Circular-orbit energy for a test-particle in Schwarzschild:**

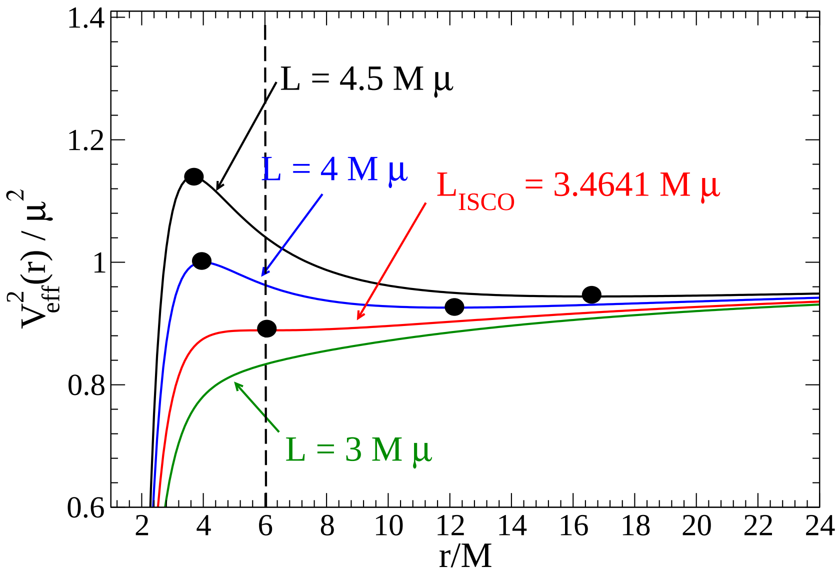
$$E_{\rm circ} = \mu \frac{1-2M/r}{\sqrt{1-3M/r}} \qquad\qquad E_{\rm circ}^{\rm PN} = \mu - \frac{\mu M}{2r}\left[1 - \frac{3}{4}\frac{M}{r} - \frac{27}{8}\frac{M^2}{r^2} - \frac{675}{64}\frac{M^3}{r^3} \cdots \right]$$

miminum of $E_{\rm circ}$ **gives ISCO** $\Rightarrow r_{\rm ISCO} = 6M$

$$E_{\rm circ}^{\rm 1PN} \Rightarrow r_{\rm ISCO}^{\rm 1PN} = 1.5M$$
$$E_{\rm circ}^{\rm 2PN} \Rightarrow r_{\rm ISCO}^{\rm 2PN} = 4.019M$$
$$E_{\rm circ}^{\rm 3PN} \Rightarrow r_{\rm ISCO}^{\rm 3PN} = 5.104M$$
$$E_{\rm circ}^{\rm 4PN} \Rightarrow r_{\rm ISCO}^{\rm 4PN} = 5.572M$$
$$E_{\rm circ}^{\rm 5PN} \Rightarrow r_{\rm ISCO}^{\rm 5PN} = 5.788M$$
$$E_{\rm circ}^{\rm 6PN} \Rightarrow r_{\rm ISCO}^{\rm 6PN} = 5.892M$$

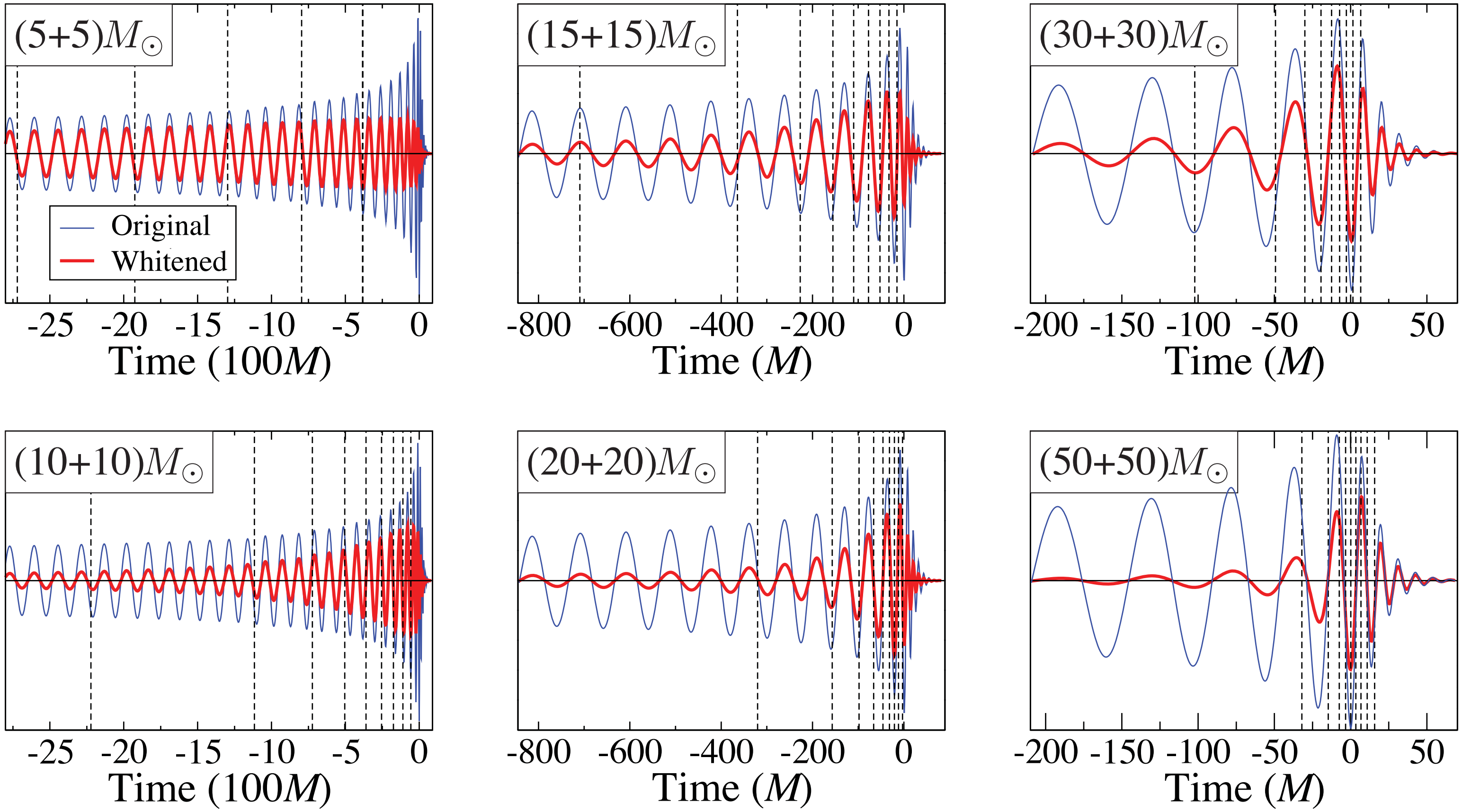
$\cdots$

$$E_{\rm circ}^{\rm 10PN} \Rightarrow r_{\rm ISCO}^{\rm 10PN} = 5.992M$$

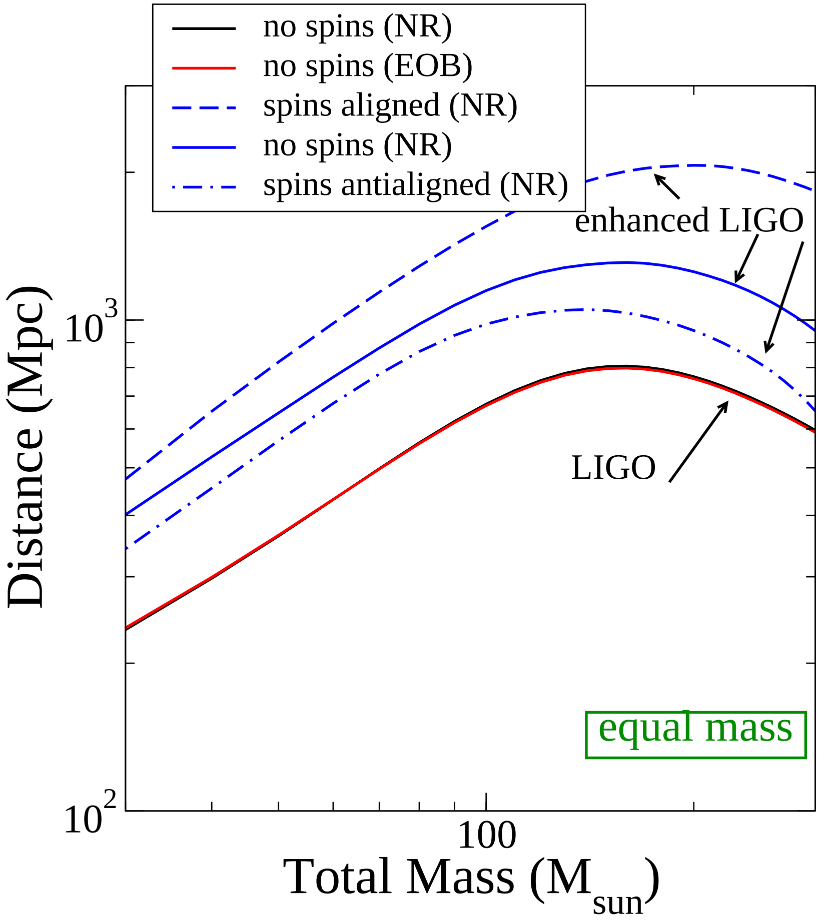


The significance of inspiral, merger and ringdown signals for LIGO

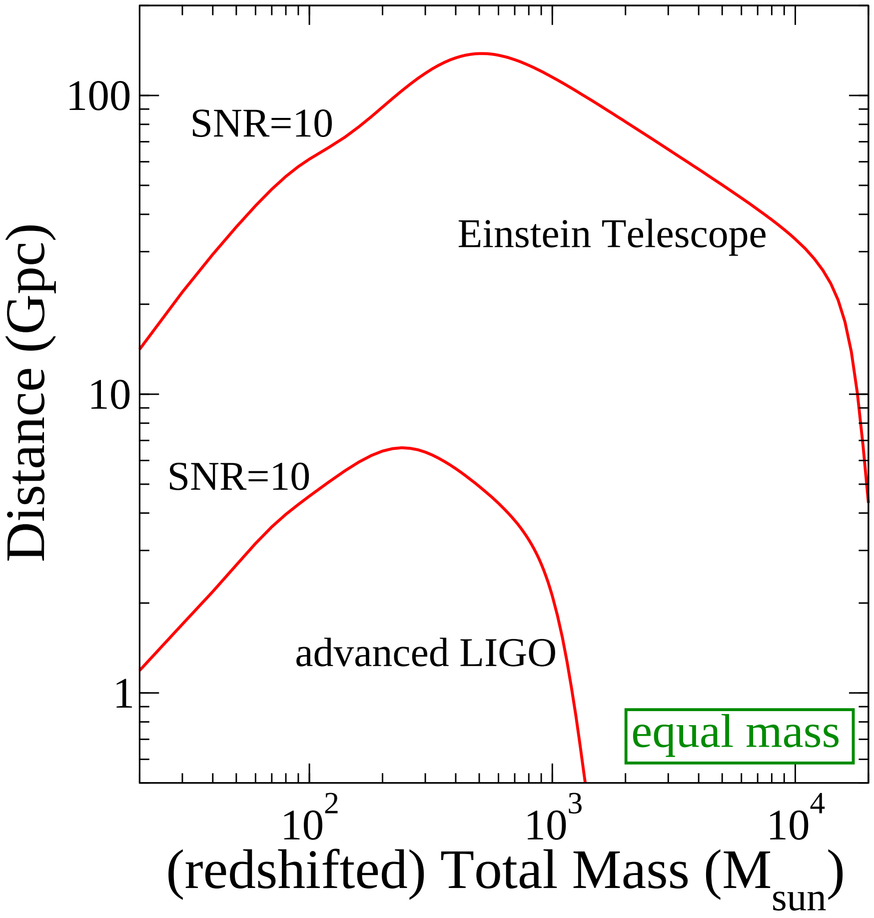
`[Pan, AB, Pretorius & NASA-Goddard 07]`



Inclusion of merger–ringdown: implications for LIGO detectors

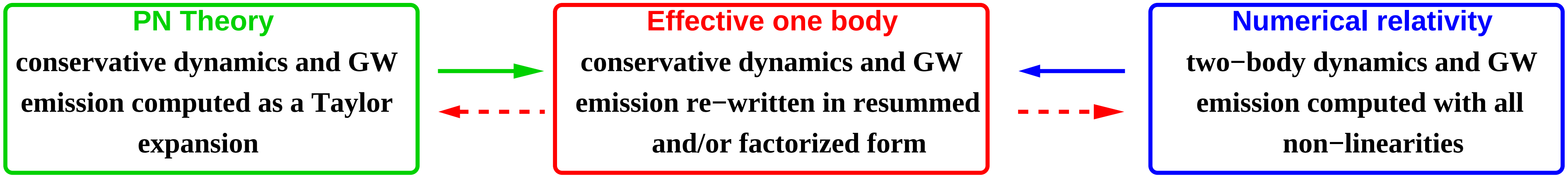


[courtesy of Pan]



Combining post-Newtonian and numerical-relativity results: the effective-one-body (EOB) approach

- **EOB approach introduced before NR breakthrough** `[AB & Damour 99, 00]`



PN Theory
conservative dynamics and GW emission computed as a Taylor expansion

Effective one body
conservative dynamics and GW emission re−written in resummed and/or factorized form

Numerical relativity
two−body dynamics and GW emission computed with all non−linearities

- **The EOB formalism uses the best information available in PN theory, but** *resums* **it in a** *suitable* **way to be able to describe accurately the full evolution: inspiral, merger and ringdown.**

- **The EOB formalism can also extract higher-order PN terms and non-perturbative effects in numerical-relativity simulations.**

Key idea for building the effective-one-body approach

- **PN-expanded Hamiltonian in the center of mass:**

$$H = H_{\rm Newt} + \frac{1}{c^2}\,H_{\rm 1PN} + \frac{1}{c^4}\,H_{\rm 2PN} + \cdots$$

The Newtonian approximation $H_{\rm Newt}$ describes a test-particle of mass μ orbiting around an *external mass* GM

- **The EOB approach is a general relativistic generalization of the forgoing fact.** **It builds an** *effective (or external) spacetime geometry* $g_{\mu\nu}^{\rm eff}(x^\alpha; GM)$ **such that the dynamics of a test-particle of mass μ moving in $g_{\mu\nu}^{\rm eff}(x^\alpha; GM)$ is** *equivalent* **to the original PN-expanded dynamics.**

- **The equivalence between the two dynamics can be thought as the equivalence between the energy spectra (quantized** *à la* **Wheeler)**

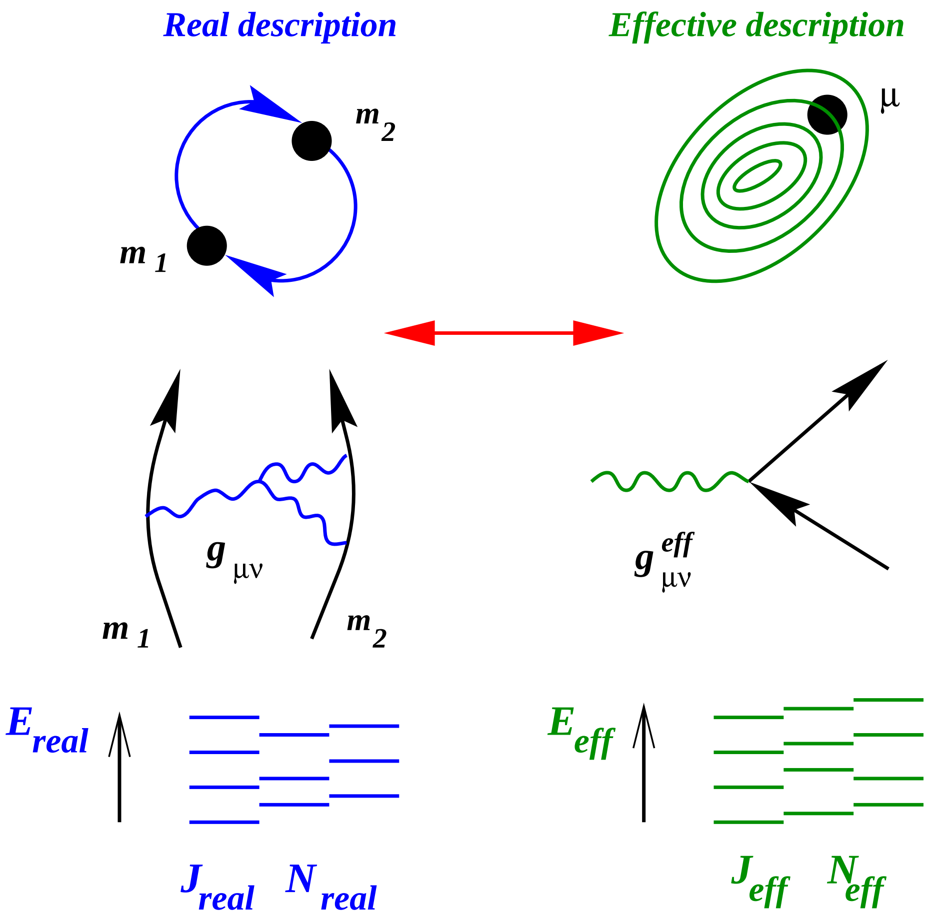
$$E_{\rm real}(N, J) = f\,[E_{\rm eff}(N, J)]$$

Effective-one-body approach in a nutshell

[AB & Damour 99]

$$\nu = \mu/M = m_1\, m_2/M^2$$
$$0 \leq \nu \leq 1/4$$

- **Resum so that *known* test-mass limit results are recovered**
- **Resum the PN expansion assuming that the equal-mass limit is a ν-deformation of the test-mass limit**



$$E_{\rm real}(N,J) = f[E_{\rm eff}(N,J)]$$

Finding the energy for *comparable-mass black holes*

- *Thinking quantum mechanically*: **the classical Hamiltonian and bound orbits are replaced by the Hamiltonian operator and quantum bound states.**

- **Real description:**

$$E_{\rm real}(N,J) = M\,c^2 - \frac{1}{2}\frac{\mu\,\alpha^2}{N^2}\left[1 + \frac{\alpha^2}{c^2}\left(\frac{6}{N\,J} - \frac{1}{4}\,\frac{15-4\nu}{N^2}\right) + \cdots\right]\,, \qquad \alpha = G\,M\,\mu$$

- **Effective description:**

$$E_{\rm eff}(N,J) = \mu\,c^2 - \frac{1}{2}\frac{\mu\,\alpha^2}{N^2}\left[1 + \frac{\alpha^2}{c^2}\left(\frac{C_{3,1}}{N\,J} + \frac{C_{4,0}}{N^2}\right) + \cdots\right]$$

- **Allow transformation of energy axis:**

$$E_{\rm eff}^{\rm NR} = E_{\rm real}^{\rm NR}\left[1 + \alpha_1\,\frac{E_{\rm real}^{\rm NR}}{\mu\,c^2} + \alpha_2\left(\frac{E_{\rm real}^{\rm NR}}{\mu\,c^2}\right)^2 + \cdots\right] \quad \Rightarrow \quad \alpha_1 = \frac{\nu}{2}, \quad \alpha_2 = 0$$

Energy for *comparable-mass bodies*

- **Classical gravity** `[AB & Damour 99]` (up to 3PN order)

$$E^2_{\rm real} = m_1^2 + m_2^2 + 2 m_1\, m_2 \left(\frac{E_{\rm eff}}{\mu}\right)$$

- **Quantum electrodynamics** (eikonal approximation) `[Brézin, Itzykson & Zinn-Justin 70]`

$$E^2_{\rm real} = m_1^2 + m_2^2 + 2 m_1\, m_2 \frac{1}{\sqrt{1+Z^2\, \alpha^2/(n-\epsilon_j)^2}}$$

- **Considering scattering states** ($E^2_{\rm real} = s$)

$$\varphi(s) \equiv \frac{s-m_1^2-m_2^2}{2 m_1\, m_2} = \frac{-(\mathbf{p}_1+\mathbf{p}_2)^2-m_1^2-m_2^2}{2 m_1\, m_2} = -\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{m_1\, m_2}$$

In summary, here is the *resummed* PN conservative dynamics

[AB & Damour 99]

"Effective" description

"Real" description

$$H^{\rm PN}_{\rm real} = H_{\rm Newt} + \frac{1}{c^2}\, H_{\rm 1PN} + \frac{1}{c^4} H_{\rm 2PN} + \cdots$$

$$H^{\nu}_{\rm eff} = \mu \sqrt{A_\nu(r)\, \left[1 + \frac{\mathbf{p}^2}{\mu^2} + \left(\frac{1}{B_\nu(r)} - 1\right)\, \frac{p_r^2}{\mu^2}\right]}$$

$$\boxed{H^{\rm EOB}_{\rm real} = M \sqrt{1 + 2\nu\, \left(\frac{H^{\nu}_{\rm eff}}{\mu} - 1\right)}}$$

$$ds^2_{\rm eff} = -A_\nu(r)\, dt^2 + B_\nu(r)\, dr^2 + r^2\, d\Omega^2$$

- **Dynamic condensed in** $A_\nu(r)$ **and** $B_\nu(r)$
- $A_\nu(r)$**, which encodes the energetics for circular orbits, is rather** *simple*

$$A_\nu(r) = 1 - \frac{2M}{r} + \frac{2M^3\nu}{r^3} + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right) \frac{M^4\,\nu}{r^4} + \frac{a_5(\nu)}{r^5} + \frac{a_6(\nu)}{r^6} + \cdots$$

If perturbed, black holes *ring or vibrate*: quasi-normal modes

[Vishveshwara 70; Press 71; Chandrasekhar et al. 75; Ferrari & Mashoon 84; Schutz & Will 85]

- If black hole *size* is $R_{\rm BH} = 2GM/c^2$,

  and $M = 1M\odot \Rightarrow R_{\rm BH} = 3\,{\rm km}$

  Travel time of spacetime *vibration*

  $\Rightarrow R_{\rm BH}/c = 10^{-2}\,{\rm msec}$

- Frequency and decay time of quasi-normal modes depend *only* on BH mass and spin

- For each $(l, m) \Rightarrow$ infinite tower of overtones

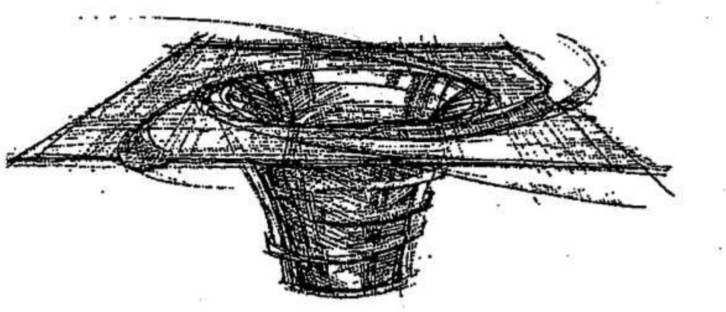
- If black hole has mass $M = 20M_\odot$, S $= 0$:

  $\omega_{200} = 604$ Hz, $\tau_{200} = 1.10$ msec

  $\omega_{201} = 560$ Hz, $\tau_{201} = 0.36$ msec

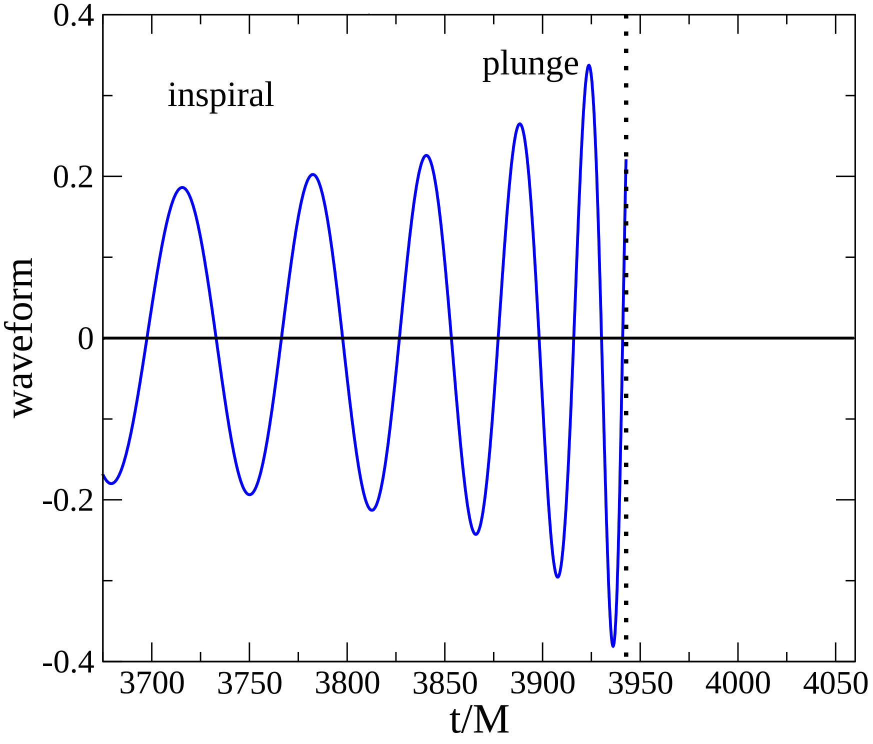
  $\omega_{202} = 486$ Hz, $\tau_{202} = 0.20$ msec

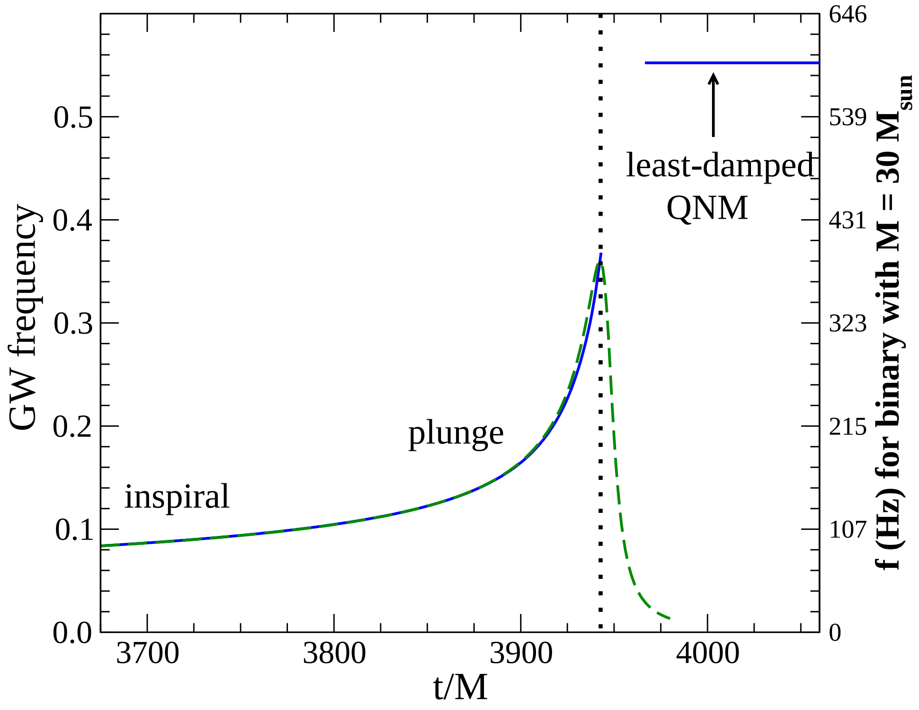
  $\ldots \qquad \ldots$



- Perturbed black-hole spacetime is intrinsically dissipative

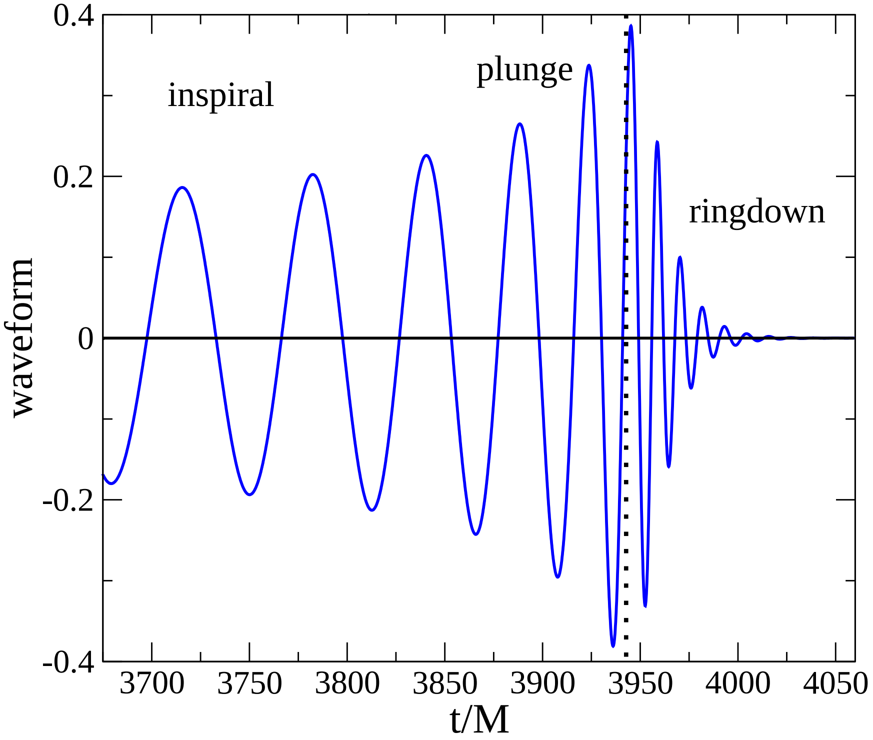
EOB inspiral-plunge waveform

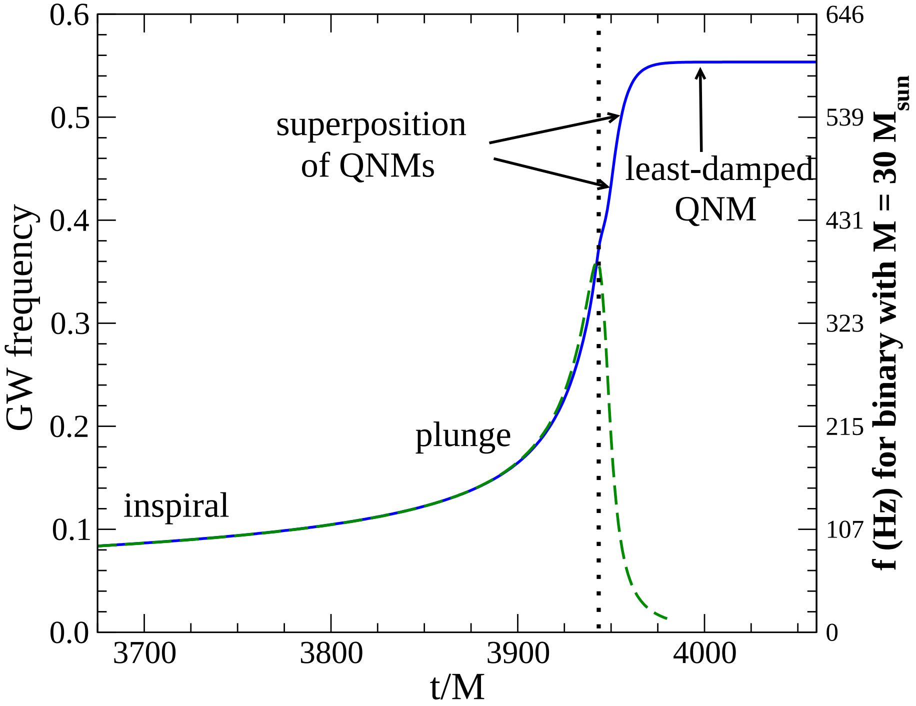




- **The plunge is a smooth continuation of the adiabatic inspiral** `[AB & Damour 00]`

EOB inspiral-merger-ringdown waveforms



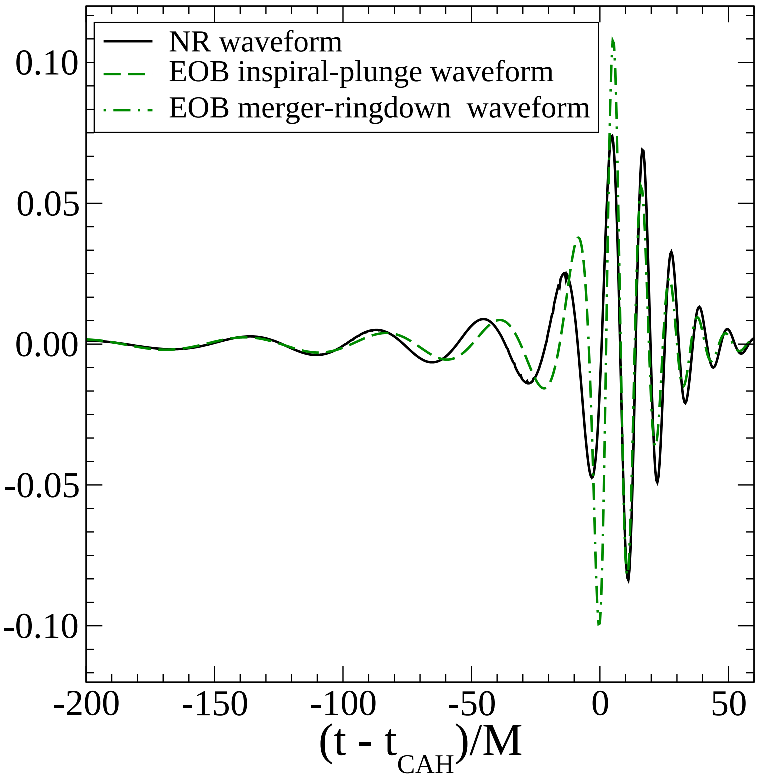


- **Very *short* transition merger–ringdown**
- **Energy quickly released during merger**
- $E_{\rm rad} \sim 2\%–12\%\, M\, c^2$

  $1 M_\odot\, c^2 \sim 10^{54}\,{\rm erg} \sim 10^{56}\,{\rm GeV!}$

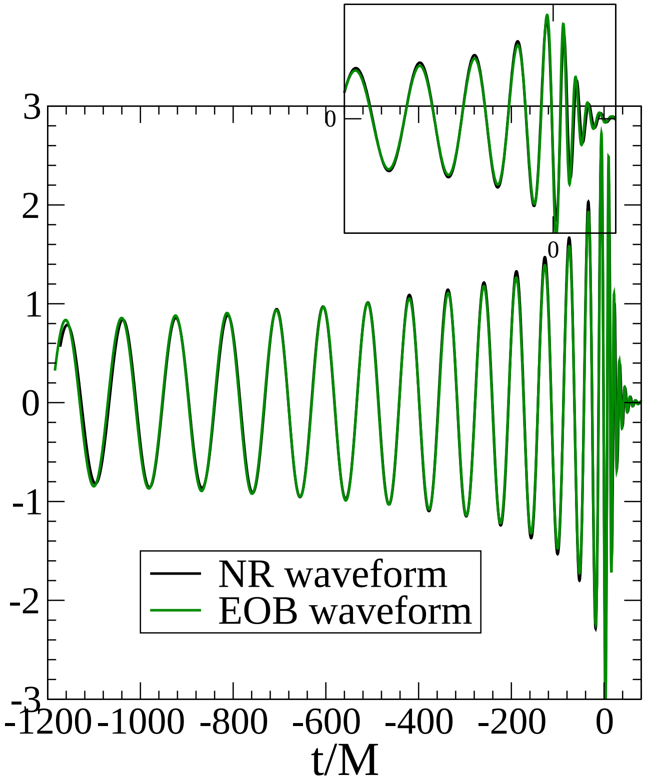
First comparisons/calibrations between NR and EOB model

`[AB, Cook & Pretorius 06]`



Uncalibrated **EOB model at 3PN order**

`[AB, Pan & NASA-Goddard 07]`

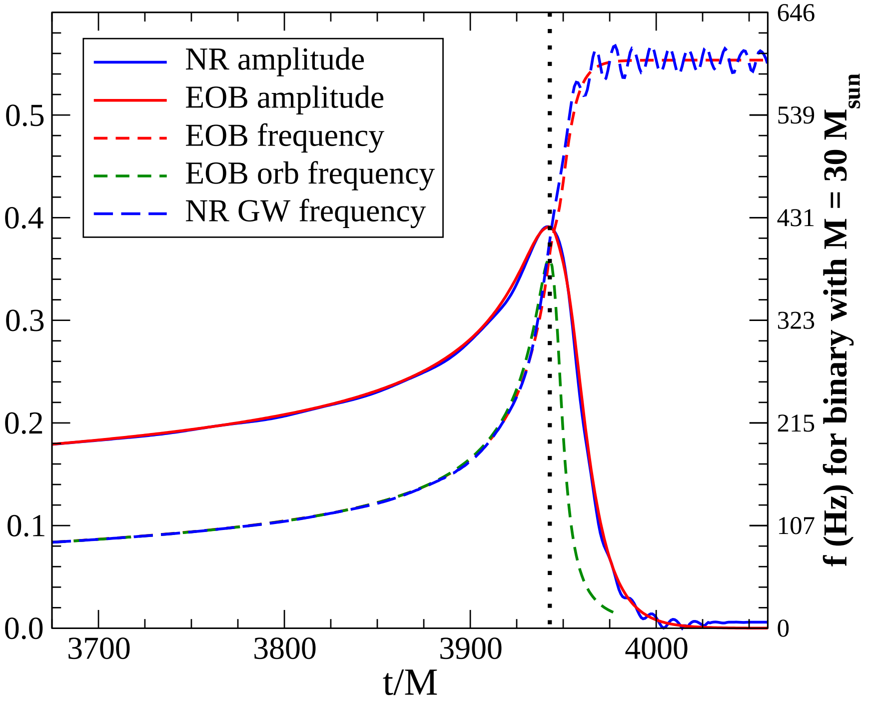


Calibrated **EOB model at p4PN order**

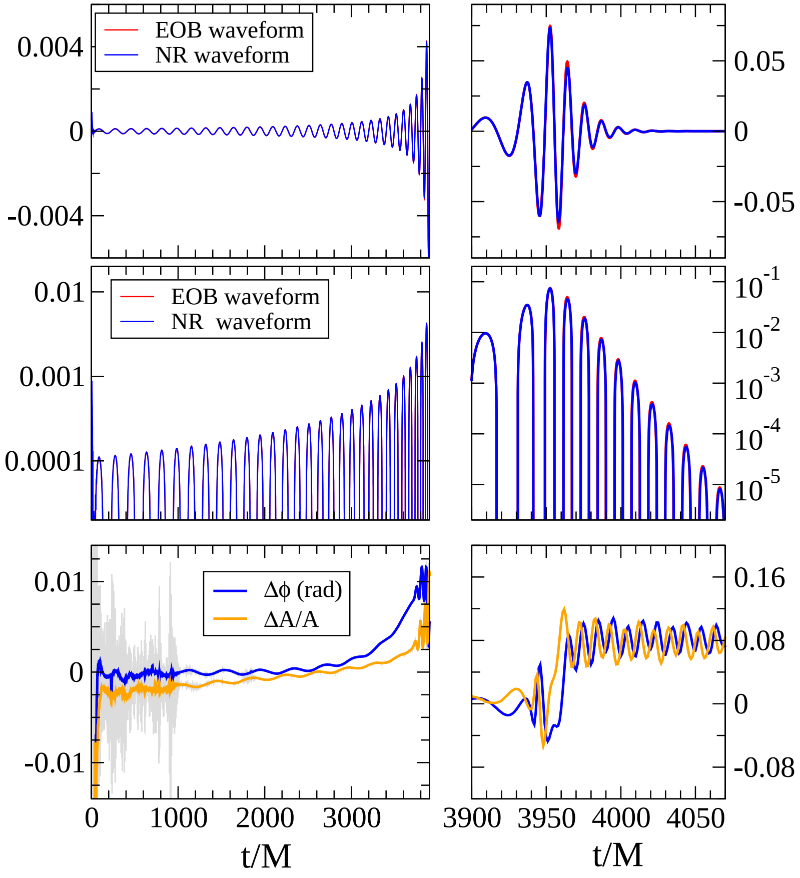
[coded in LIGO software by Ochsner & Pan]

Most recent calibrations using an extremely accurate simulation

`[AB, Pan, Pfeiffer, Scheel, Buchman & Kidder 09]`

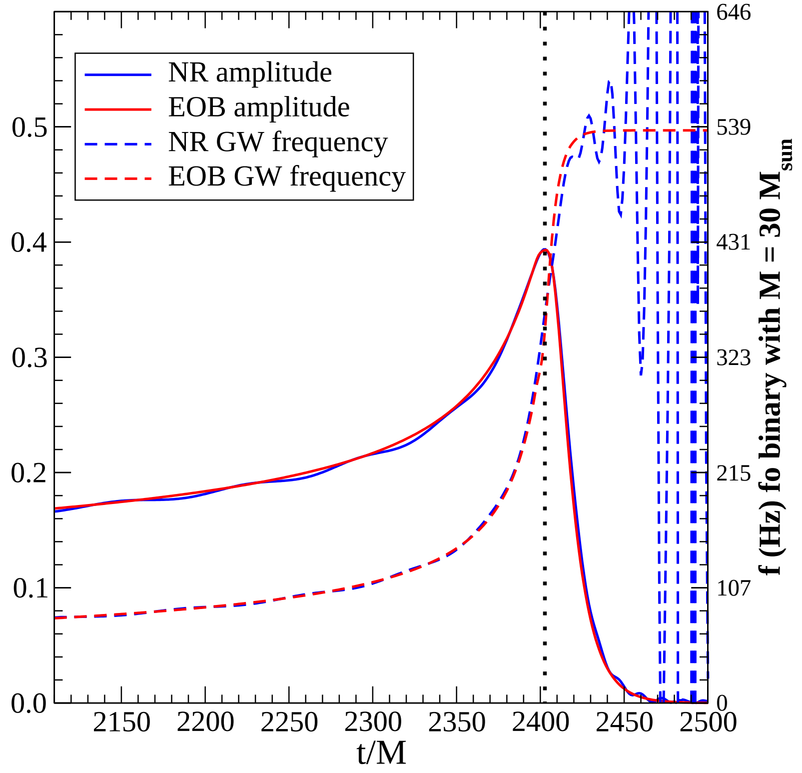


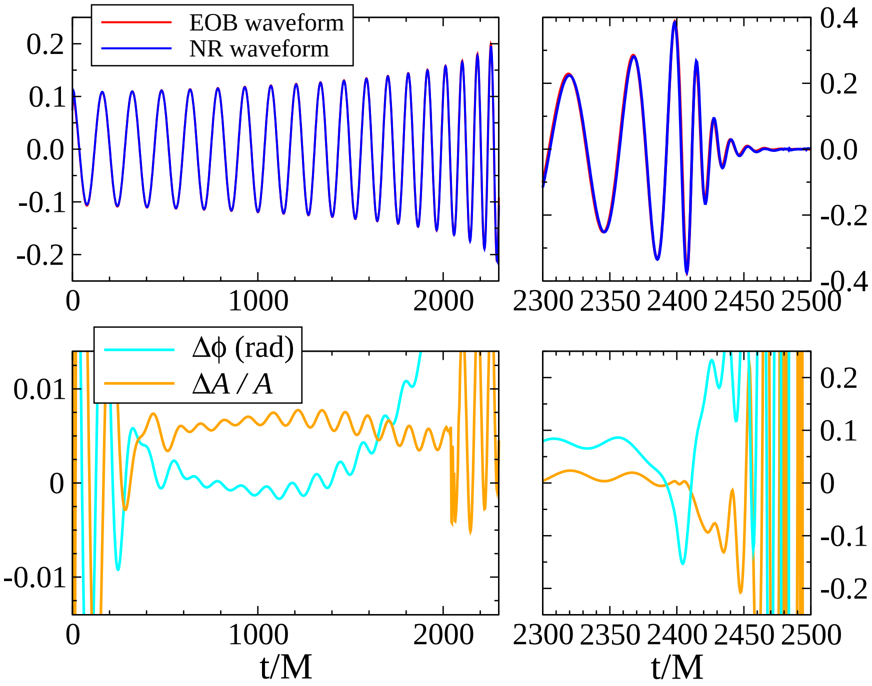
see also `[Damour & Nagar 09]`



Calibrations for a spinning, non-precessing black-hole binary

`[Pan, AB & Caltech-Cornell; Pan, AB, Fujita, Racine & Tagoshi (in preparation)]`





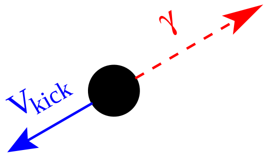
Building on recent work `[Barausse, Racine & AB 09]` ⇒ **spin EOB model** `[Barausse & AB]`

Gravitational recoil *or kick*

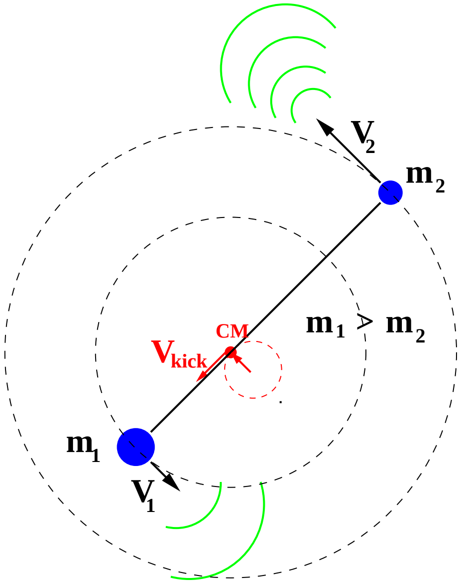
[Peres 62; Bekenstein 73; Fitchett 83; Fitchett & Detweiler 84; Wiseman 92; Kidder 95; Blanchet et al. 05]

[Damour & Gopakumar 06; Racine, AB & Kidder 08]

- **In atomic physics, *free* atom recoils when photon is emitted because of momentum conservation.**



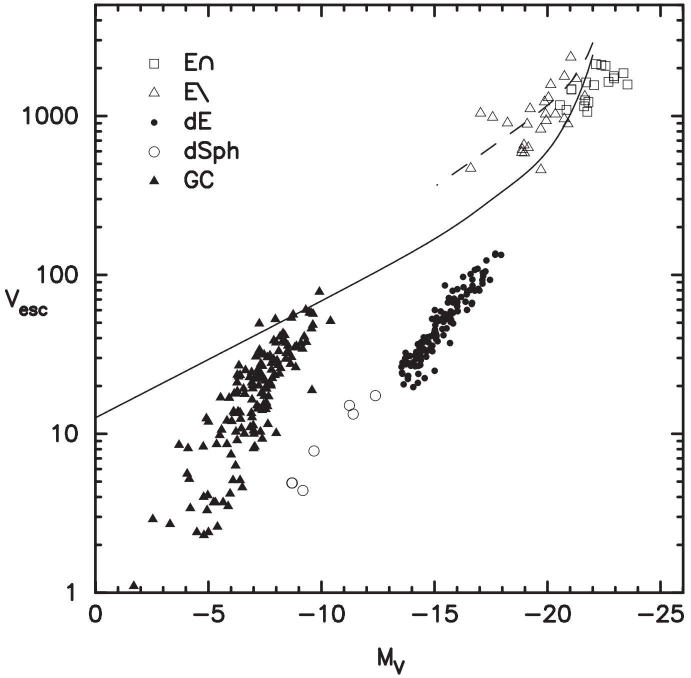
- **In general relativity:** $dP_{\rm CM}/dt = -dP_{\rm GW}/dt$



$$V_{\rm kick}/c \sim \nu^2 \sqrt{1-4\nu}\, \left(\frac{2M}{r}\right)^4$$

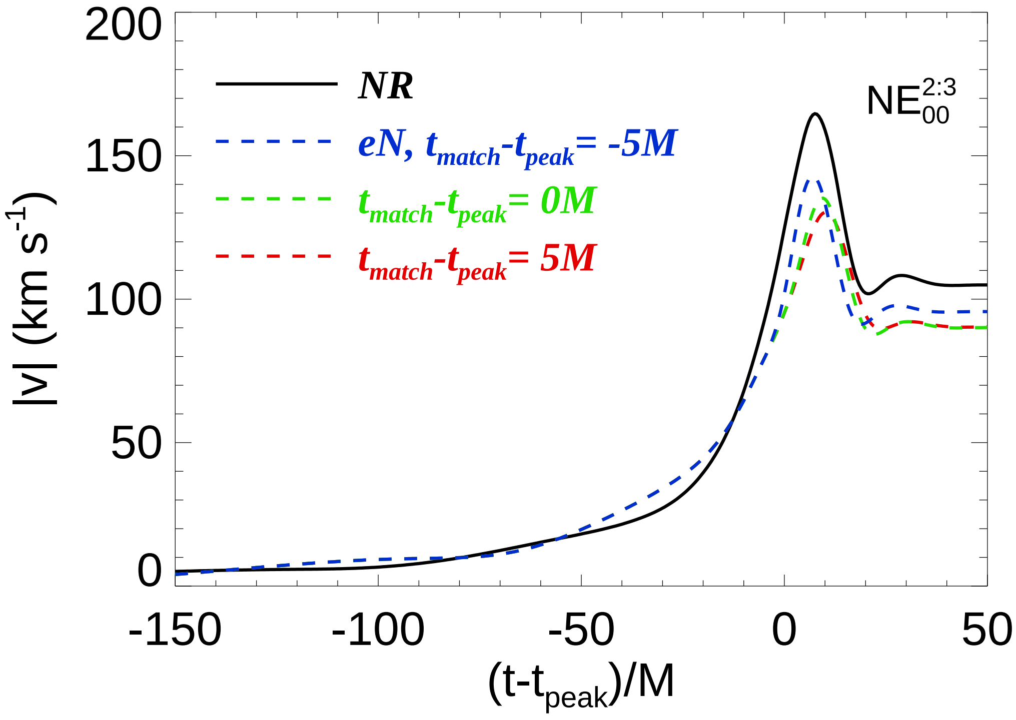
- **To produce net recoil we need *asymmetry* in mass and/or spins, and inspiralling motion**

[Merritt et al. 04]



Analytical modeling of the kick

[Schnittman, AB & NASA-Goddard 07]



Analytical model describing merger and ringdown as superposition of QNMs

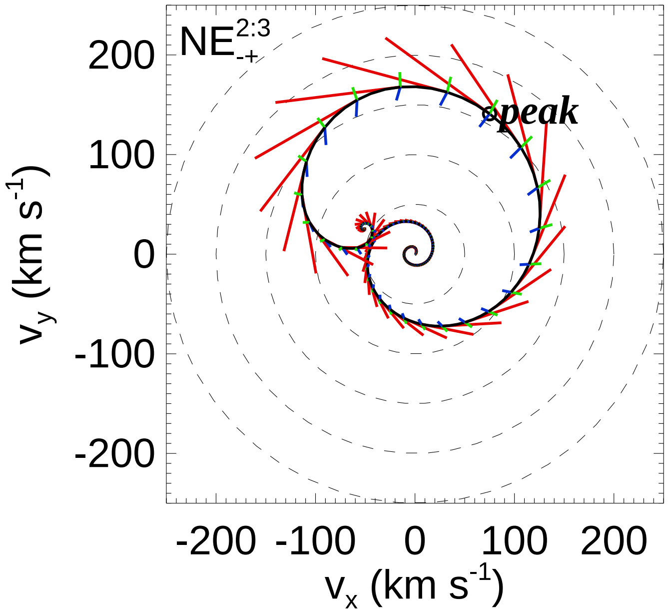
How the kick builds up throughout inspiral, merger and ringdown

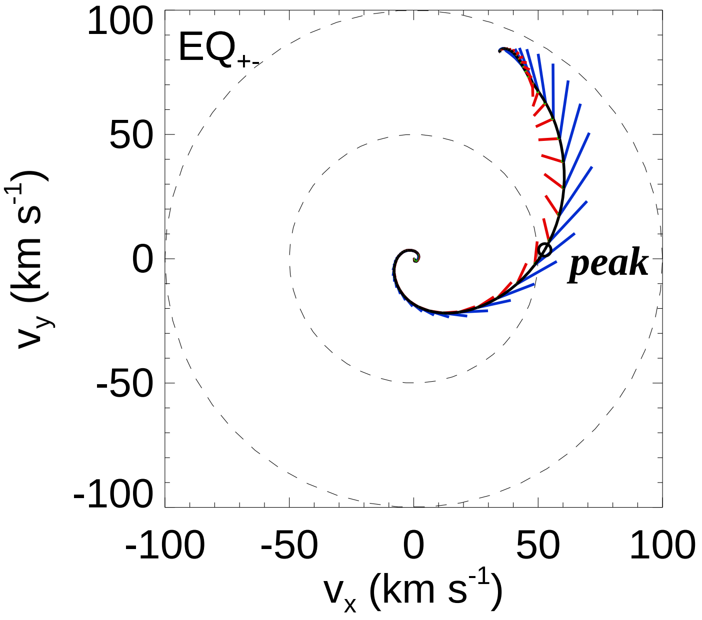
[Schnittman, AB & NASA-Goddard 07]

$I^{22} \Rightarrow$ mass quadr., $I^{33} \Rightarrow$ mass octup.

$S^{21} \Rightarrow$ current quadr., $I^{44} \Rightarrow$ mass hexad.

$$|\mathbf{V}_{\rm kick}| \simeq \int \left[\hat{\mathbf{V}} \cdot \frac{d\mathbf{P}}{dt}(I^{22}\, S^{21}) + \hat{\mathbf{V}} \cdot \frac{d\mathbf{P}}{dt}(I^{22}\, I^{33}) + \hat{\mathbf{V}} \cdot \frac{d\mathbf{P}}{dt}(I^{33}\, I^{44}) \right] dt$$





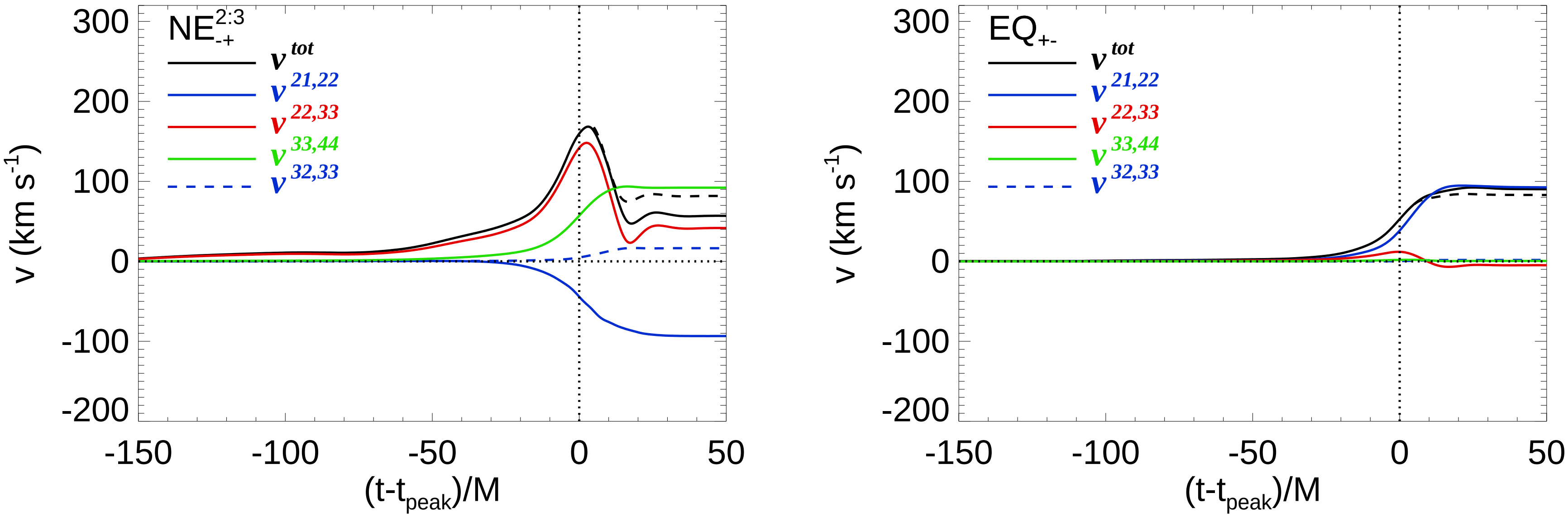
Anatomy of the kick and anti-kick

`[Schnittman, AB & NASA-Goddard 07]`

- **Magnitude of anti-kick depends on QNM-frequencies associated with dominant modes**

$I^{22}\, I^{33\,*}$: $(\omega_{33}^{\rm QNM} - \omega_{22}^{\rm QNM})$ **is large** $\qquad$ $I^{22\,*}\, S^{21}$: $(\omega_{21}^{\rm QNM} - \omega_{22}^{\rm QNM})$ **is small**

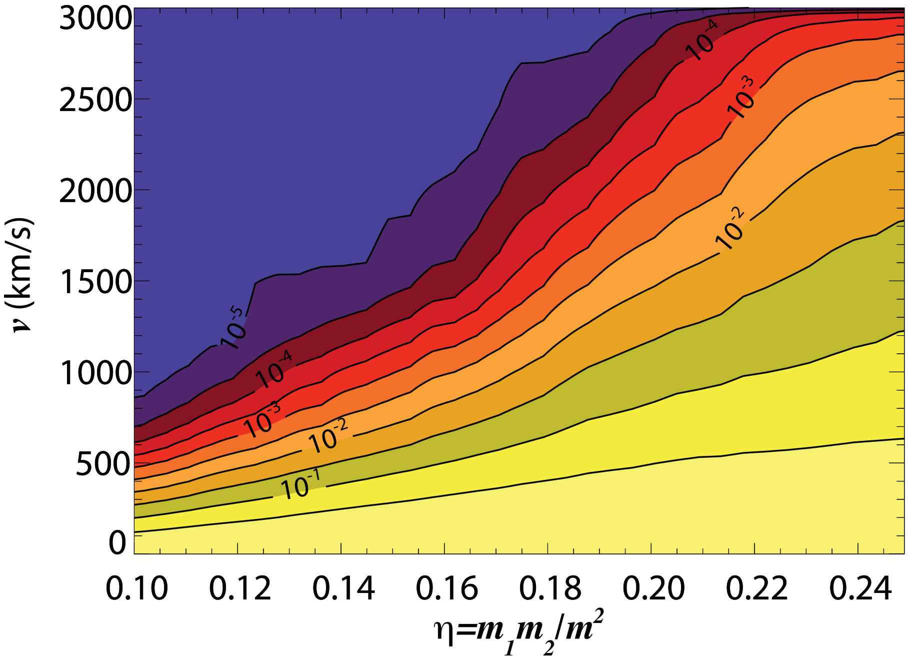
$\Rightarrow$ **spiral back inward** $\qquad$ $\Rightarrow$ **drifts off**



Distribution of gravitational kicks using analytical modeling

- **Using a first example of the EOB model with spins** [Schnittman & AB 07]

Random spins, $a_1 = a_2 = 0.9$, $1/10 < m_2/m_1 < 1$



$$f_{v_{\rm kick}>500} = 0.12^{+0.06}_{-0.05}$$

$$f_{v_{\rm kick}>1000} = 0.027^{+0.021}_{-0.014}$$

see also [Campanelli et al. 07, 09; Herrmann et al. 07; Baker et al. 08; Lousto et al. 08]

Conclusions

- **The detection of coalescing BHs will open an exciting new era for astronomy.**
- **We need to prepare the** *best tools* **to be able to extract the** *best science* **in astrophysics, general relativity or fundamental physics.**
- **Two-body problem in Newtonian gravity is far more simple than in GR.**
- **It is possible to model analytically and** *understand* **the coalescence of binary BHs by:** *resumming* **the PN-expansion during inspiral and plunge; using insights from BH perturbation theory for the ringdown, and results from NR for the merger.**
- **There is an intriguing** *simplicity* **to binary coalescence: details of merger hidden behind the curvature potential barrier, eventually swallowed by the newborn BH.**
- **EOB approach condenses dynamics and GW emission in a few crucial functions. It can be easily calibrated to build highly accurate, full waveforms for spinning, precessing BH binaries for Advanced LIGO and LISA.**
- **Merging BHs can get** *kicks* **which can also be modeled and** *understood* **analytically.**

Collaborators

- **University of Maryland:**

  Postdocs: *Enrico Barausse, Yi Pan, Etienne Racine (now at the Canadian Space Agency), Jeremy Schnittman (now at Johns Hopkins)*

  Graduate students: *Evan Ochsner, Andrea Taracchini*

- **NASA Goddard Flight Space Center:** *Joan Centrella, John Baker, Bernard Kelly Sean McWilliams, Jim Van Meter*

- **Caltech/Cornell:** *Luisa Buchman, Larry Kidder, Harald Pfeiffer, Mark Scheel*

- **Princeton University:** *Frans Pretorius*

- **Wake Forest University:** *Greg Cook*

- **Institut d'Astrophysique de Paris:** *Luc Blanchet, Guillaume Faye*