Jets, Kinematics, and Other Variables

A Tutorial for Physics With p-p (LHC/Cern) and p-\(\bar{p}\) (Tevatron/FNAL) Experiments

*Drew Baden*

*University of Maryland*

Elastic scattering
Forward-forward scattering, no disassociation (protons stay protons)

\[ b \gg 2 \, r_p \]
“Single-diffractive” scattering

One of the 2 nucleons disassociates into a spray of particles

- Mostly $\pi^\pm$ and $\pi^0$ particles
- Mostly in the forward direction following the parent nucleon’s momentum

\[ b \sim 2 r_p \]
"Double-diffractive" scattering

Both nucleons break up

Resultant spray of particles is in the forward direction

$b < r_p$

Active detector

Active detector
Proton-(anti)Proton Collisions

- At “high” energies we are probing the nucleon structure
  - “High” means Compton wavelength $\lambda_{\text{beam}} \equiv \frac{\hbar c}{E_{\text{beam}}} \ll r_{\text{proton}} \sim \frac{\hbar c}{1\text{GeV}} \sim 1\text{fm}$
    - $E_{\text{beam}}=1\text{TeV}\text{@FNAL} \quad 5-7\text{ TeV}\text{@LHC}$
  - We are really doing parton–parton scattering ($\text{parton} = \text{quark, gluon}$)

- Look for scatterings with large momentum transfer, ends up in detector “central region” (large angles wrt beam direction)
  - Each parton has a momentum distribution –
    - CM of hard scattering is not fixed as in $e^+e^-$ will be move along z-axis with a boost
    - This motivates studying boosts along z
  - What’s “left over” from the other partons is called the “underlying event”

- If no hard scattering happens, can still have disassociation
  - An “underlying event” with no hard scattering is called “minimum bias”
By far most of the processes in nucleon-nucleon scattering are described by:

- \( \sigma(\text{Total}) \sim \sigma(\text{scattering}) + \sigma(\text{single diffractive}) + \sigma(\text{double diffractive}) \)

This can be naively estimated…

- hard sphere scattering, partial wave analysis:
  - \( \sigma \sim 4 \times \text{Area}_{\text{proton}} = 4\pi r_p^2 = 4\pi \times (1 \text{fm})^2 \sim 125 \text{mb} \)

But! total cross-section stuff is NOT the reason we do these experiments!

Examples of “interesting” physics @ Tevatron
- \( W \) production and decay via lepton
  - \( \sigma \cdot \text{Br}(W \rightarrow e\nu) \sim 2 \text{nb}, 1 \text{ in } 50 \times 10^6 \text{ collisions} \)
- \( Z \) production and decay to lepton pairs
  - About 1/10 that of \( W \) to leptons
- Top quark production
  - \( \sigma(\text{total}) \sim 5 \text{pb}, 1 \text{ in } 20 \times 10^9 \text{ collisions} \)

Rates for similar things at LHC will be \( \sim 10 \times \) higher
• What determines number of detected events $N(X)$ for process “$X$”?  
  – Or the rate: $R(X) = N(X)/\text{sec}$?  
• $N(X)$ per unit cross-section should be a function of the brightness of the beams  
  – And should be constant for any process:  
    $N(X)/\sigma(X) = \text{constant} = L$ (luminosity)  
    $R(X)/\sigma(X) = L$ (instantaneous luminosity)  
• Units of luminosity:  
  – “Number of events per barn”  
  – Note: $1\text{nb} = 10^{-9} \text{ barns} = 10^{-9} \times 10^{-24} \text{ cm}^2 = 10^{-33} \text{ cm}^2$  
  – LHC instantaneous design luminosity  
    $10^{34} \text{ cm}^{-2} \text{ s}^{-1} = 10 \text{ nb}^{-1} / \text{s},$ or 10 events per nb cross-section per second, or “10 inverse nanobarns per second”  
  • e.g. 10 t-tbar events per second
Coordinates

Proton beam direction

Proton or anti-proton beam direction

Detector

x

y

z

θ

r

φ
Detect the “hard scattering”
Phase Space

- **Relativistic invariant phase-space element:**
  - Define $p\bar{p}$ or $pp$ collision axis along $z$-axis:
  - Coordinates $p^\mu = (E, p_x, p_y, p_z)$ – Invariance with respect to boosts along $z$?
    - 2 longitudinal components: $E$ & $p_z$ (and $dp_z/E$) NOT invariant
    - 2 transverse components: $p_x, p_y$, (and $dp_x, dp_y$) ARE invariant

- **Boosts along $z$-axis**
  - For convenience: define $p^\mu$ where only 1 component is not Lorentz invariant
  - Choose $p_T, m, \phi$ as the “transverse” (invariant) coordinates
    - $p_T = p \sin(\theta)$ and $\phi$ is the azimuthal angle
  - For 4th coordinate define “rapidity” ($y$)
    - $y \equiv \frac{1}{2} \ln \frac{E + p_z}{E - p_z}$ or $p_z = E \tanh y$
    - …How does it transform?
Boosts Along beam-axis

- Form a boost of velocity $\beta$ along $z$ axis
  - $p_z \Rightarrow \gamma(p_z + \beta E)$
  - $E \Rightarrow \gamma(E + \beta p_z)$
  - Transform rapidity:

\[
y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} \Rightarrow \frac{1}{2} \ln \frac{\gamma(E + \beta p_z) + \gamma(p_z + \beta E)}{\gamma(E + \beta p_z) - \gamma(p_z + \beta E)}
\]
\[
= \frac{1}{2} \ln \left( \frac{E + p_z}{E - p_z} \right)^{(1 + \beta)} = y + \ln \gamma(1 + \beta)
\]
\[
y \Rightarrow y + y_b
\]

- Boosts along the beam axis with $v=\beta c$ will change $y$ by a constant $y_b$
  - $(p_T,y,\phi,m) \Rightarrow (p_T,y+y_b,\phi,m)$ with $y \Rightarrow y + y_b$, $y_b = \ln \gamma(1+\beta)$ simple additive to rapidity

- Relationship between $y$, $\beta$, and $\theta$ can be seen using $p_z = p \cos(\theta)$ and $p = \beta E$

\[
y = \frac{1}{2} \ln \frac{1 + \beta \cos \theta}{1 - \beta \cos \theta}
\]
or
\[
tanh y = \beta \cos \theta
\]
where $\beta$ is the CM boost
Phase Space (cont)

- Transform phase space element $d\tau$ from $(E,p_x,p_y,p_z)$ to $(p_t, y, \phi, m)$
  \[
  dp_x dp_y = \frac{1}{2} dp_T^2 d\phi
  \]

- Gives:
  \[
  d\tau = \frac{1}{2} dp_T^2 d\phi dy
  \]

- Basic quantum mechanics: $d\sigma = |M|^2 d\tau$
  - If $|M|^2$ varies slowly with respect to rapidity, $d\sigma/dy$ will be \(~constant\) in $y$
  - Origin of the “rapidity plateau” for the min bias and underlying event structure
  - Apply to jet fragmentation - particles should be uniform in rapidity wrt jet axis:
    - We expect jet fragmentation to be function of momentum perpendicular to jet axis
    - This is tested in detectors that have a magnetic field used to measure tracks

\[
\begin{align*}
  dy & = dp_z \left( \frac{\partial y}{\partial p_z} + \frac{\partial y}{\partial E} \frac{\partial E}{\partial p_z} \right) \\
  &= dp_z \left( \frac{E}{E^2 - p_z^2} - \frac{p_z^2}{E^2 - p_z^2} \frac{p_z}{E} \right) \\
  &= \frac{dp_z}{E}
\end{align*}
\]

\[
y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z}
\]
Transverse Energy and Momentum Definitions

- Transverse Momentum: momentum perpendicular to beam direction:
  \[ p_T^2 = p_x^2 + p_y^2 \quad \text{or} \quad p_T = p \sin \theta \]

- Transverse Energy defined as the energy if \( p_z \) was identically 0:
  \[ E_T \equiv E(p_z=0) \]
  \[ E_T^2 = p_x^2 + p_y^2 + m^2 = p_T^2 + m^2 = E^2 - p_z^2 \]

- How does \( E \) and \( p_z \) change with the boost along beam direction?
  
  - Using \( \tanh y = \beta \cos \theta \) and \( p_z = p \cos \theta \) gives \( p_z = E \tanh y \)

  then \( E_T^2 = E^2 - p_z^2 = E^2 - E^2 \tanh^2 y = E^2 \text{sech}^2 y \)

  or \( E = E_T \cosh y \) which also means \( p_z = E_T \sinh y \)

  - (remember boosts cause \( y \rightarrow y + y_b \))
  - Note that the sometimes used formula \( E_T = E \sin \theta \) is not (strictly) correct!
  - But it’s close – more later….
Invariant Mass $M_{1,2}$ of 2 particles $p_1$, $p_2$

- Well defined: $M_{1,2}^2 = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2(E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2)$

- Switch to $p^\mu=(p_T,y,\phi,m)$ (and do some algebra…)

$$\vec{p}_1 \cdot \vec{p}_1 = p_{x_1} p_{x_2} + p_{y_1} p_{y_2} + p_{z_1} p_{z_2} = E_T E_T \left( \beta_T \beta_T \cos \Delta \phi + \sinh y_1 \sinh y_2 \right)$$

with $E = E_T \cosh y$ and $\beta_T \equiv p_T / E_T$

- This gives $M_{1,2}^2 = m_1^2 + m_2^2 + 2E_T E_T \left( \cosh \Delta y - \beta_T \beta_T \cos \Delta \phi \right)$
  - With $\beta_T \equiv p_T / E_T$
  - Note:
    - For $\Delta y \to 0$ and $\Delta \phi \to 0$, high momentum limit: $M \to 0$: angles “generate” mass
    - For $\beta \to 1$ ($m/p \to 0$) $M_{1,2}^2 = 2E_T E_T \left( \cosh \Delta y - \cos \Delta \phi \right)$

This is a useful formula when analyzing data…
Invariant Mass, multi particles

- Extend to more than 2 particles:

\[ M_{1,2,3}^2 = (p_1 + p_2 + p_3)^2 = (p_1 + p_2)^2 + 2(p_1 + p_2)p_3 + m_3^2 \]
\[ = M_{1,2}^2 + [2p_1p_3] + [2p_2p_3] + m_3^2 \]
\[ = M_{1,2}^2 + [p_1^2 + 2p_1p_3 + p_3^2] - m_1^2 - m_3^2 + [p_2^2 + 2p_2p_3 + p_3^2] - m_2^2 - m_3^2 + m_3^2 \]
\[ = M_{1,2}^2 + M_{1,3}^2 + M_{2,3}^2 - m_1^2 - m_2^2 - m_3^2 \]

- In the high energy limit as \( m/p \rightarrow 0 \) for each particle:

\[ M_{1,2,3}^2 = M_{1,2}^2 + M_{2,3}^2 + M_{1,3}^2 \]

⇒ Multi-particle invariant masses where each mass is negligible – no need to id

⇒ Example: \( t \rightarrow Wb \) and \( W \rightarrow \text{jet+jet} \)
- Find \( M(\text{jet,jet,b}) \) by just adding the 3 2-body invariant masses in quadriture
- Doesn’t matter which one you call the b-jet and which the “other” jets as long as you are in the high energy limit
Pseudo-rapidity
"Pseudo" rapidity and "Real" rapidity

- Definition of $y$: $\tanh(y) = \beta \cos(\theta)$
  - Can almost (but not quite) associate position in the detector ($\theta$) with rapidity ($y$)
- But...at Tevatron and LHC, most particles in the detector (>90%) are $\pi$'s with $\beta \approx 1$
- Define "pseudo-rapidity" defined as $\eta = y(\theta, \beta=1)$, or $\text{tanh}(\eta) = \cos(\theta)$ or
  $$\eta = \frac{1}{2} \ln \frac{1 + \cos \theta}{1 - \cos \theta} = \ln \frac{\cos \theta/2}{\sin \theta/2} = -\ln(\tan \theta/2)$$

($\eta=5, \theta=0.77^\circ$)
Rapidity (y) vs “Pseudo-rapidity” (\( \eta \))

- From \( \tanh(\eta) = \cos(\theta) = \tanh(y)/\beta \)
  - We see that \( |\eta| \geq |y| \)
  - Processes “flat” in rapidity \( y \) will not be “flat” in pseudo-rapidity \( \eta \)
    - (\( y \) distributions will be “pushed out” in pseudo-rapidity)
At colliders, Center-of-Mass can be moving with respect to detector frame

- Lots of longitudinal momentum can escape down beam pipe
  - But transverse momentum $p_T$ is conserved in the detector

Plot $\eta - y$ for constant $m_\pi$, $p_T \Rightarrow \beta(\theta)$

For all $\eta$ in DØ/CDF, can use $\eta$ position to give $y$:
- Pions: $|\eta| - |y| < 0.1$ for $p_T > 0.1$GeV
- Protons: $|\eta| - |y| < 0.1$ for $p_T > 2.0$GeV
- As $\beta \rightarrow 1$, $y \rightarrow \eta$ (so much for “pseudo”)
Rapidity “plateau”

- Constant $p_t$, rapidity plateau means $d\sigma/dy \sim k$
  - How does that translate into $d\sigma/d\eta$?

$$\frac{d\sigma}{d\eta} = \frac{d\sigma}{dy} \frac{dy}{d\eta} = k \frac{dy}{d\eta}$$

- Calculate $dy/d\eta$ keeping $m$, and $p_t$ constant
- After much algebra… $dy/d\eta = \beta(\eta)$

$$\frac{d\sigma}{d\eta} = \frac{d\sigma}{dy} \frac{dy}{d\eta} = k \frac{dy}{d\eta} = k\beta(\eta)$$

- “pseudo-rapidity” plateau…only for $\beta \to 1$

...some useful formulae...

$$\tanh(y) = \beta(\eta) \tanh(\eta)$$

$$\beta(\eta) = \frac{p}{E} = \frac{\sqrt{p_T^2 + p_Z^2}}{p_T^2 + p_Z^2 + m^2} = \frac{\cosh(\eta)}{\sqrt{m^2 / p_T^2 + \cosh^2(\eta)}}$$

$$\beta(\eta)$$
Measured momentum conservation

- Momentum conservation: 
  \[ \sum_{\text{particles}} p_Z = P_{CM} \quad \text{and} \quad \sum_{\text{particles}} \vec{p}_T = 0 \]

- What we measure using the calorimeter: 
  \[ \sum_{\text{cells}} p_Z = P_{CM} \quad \text{and} \quad \sum_{\text{cells}} \vec{p}_T = 0 \]

- For processes with high energy neutrinos in the final state: 
  \[ \sum_{\text{cells}} \vec{p}_T + \vec{p}_{TV} = 0 \]

- We “measure” \( p_\nu \) by “missing \( p_T \)” method: 
  \[ \vec{p}_T = \vec{p}_\nu \equiv - \sum_{\text{cells}} \vec{E}_T \]
  - e.g. \( W \rightarrow e\nu \) or \( \mu\nu \)

- Longitudinal momentum of neutrino cannot be reliably estimated
  - “Missing” measured longitudinal momentum also due to CM energy going down beam pipe due to the other (underlying) particles in the event
  - This gets a lot worse at LHC where there are multiple pp interactions per crossing
    - Most of the interactions don’t involve hard scattering so it looks like a busier underlying event
Transverse Mass

- Since we don’t measure $p_z$ of neutrino, cannot construct invariant mass of W
- What measurements/constraints do we have?
  - Electron 4-vector
  - Neutrino 2-d momentum ($p_T$) and $m=0$
- So construct “transverse mass” $M_T$ by:
  1. Form “transverse” 4-momentum by ignoring $p_z$ (or set $p_z=0$)
  2. Form “transverse mass” from these 4-vectors:

$$p_T^\mu \equiv (E_T, p_T, 0)$$

$$M_{T1,2}^2 \equiv (p_{T1} + p_{T2})^\mu (p_{T1} + p_{T2})_{\mu}$$

- This is equivalent to setting $\eta_1 = \eta_2 = 0$
- For $e/\mu$ and $\nu$, set $m_e = m_\mu = m_\nu = 0$ to get:

$$M_{T1,2}^2 = 2 E_{T1} E_{T2} (1 - \cos \Delta \phi) = 4 E_{T1} E_{T2} \sin^2 (\Delta \phi/2)$$

- This is another way to see that the opening angle “generates” the mass
Transverse Mass Kinematics for Ws

- Transverse mass distribution?
- Start with \( M_W^2 = M_{e,\nu}^2 = 2E_{Te}E_{Tv}(\cosh\Delta \eta - \cos \Delta \phi) \)

- Constrain to \( M_W=80\text{GeV} \) and \( p_T(W)=0 \)
  - \( \cos \Delta \phi = -1 \)
  - \( E_{Te} = E_{Tv} \)
  - This gives you \( E_{Te}E_{Tv} \) versus \( \Delta \eta \)

\[
E_{Te}E_{Tv} = \frac{80^2}{2(\cosh\Delta \eta + 1)}
\]

- Now construct transverse mass

\[
M_{Te,\nu}^2 = 2E_{Te}E_{Tv}(1 - \cos \Delta \phi) = 2 \frac{80^2}{\cosh \Delta \eta + 1}
\]

  - Clearly \( M_T=M_W \) when \( \eta_e=\eta_\nu=0 \)
Neutrino Rapidity

• Can you constrain $M(e, \nu)$ to determine the pseudo-rapidity of the $\nu$?
  – Would be nice, then you could veto on $\theta_\nu$ in “crack” regions

• Use $M(e, \nu) = 80$GeV and

$$M_w^2 = 80^2 = 2E_{Te} E_{Tv} \left( \cosh \Delta \eta - \cos \Delta \phi \right)$$

to get

$$\cosh \Delta \eta = \frac{80^2}{2E_{Te} E_{Tv}} + \cos \Delta \phi$$

and solve for $\Delta \eta$:

$$\Delta \eta = \ln \frac{\cosh \Delta \eta + \sqrt{\cosh^2 \Delta \eta + 1}}{2}$$

• Since we know $\eta_e$, we know that $\eta_\nu = \eta_e \pm \Delta \eta$
  – Two solutions. Neutrino can be either higher or lower in rapidity than electron
  – Why? Because invariant mass involves the opening angle between particles.
  – Perhaps this can be used for neutrino’s (or other sources of missing energy?)
Jets
Jet Definition

- How to define a “jet” using calorimeter towers so that we can use it for invariant mass calculations
  - And for inclusive QCD measurements (e.g. $d\sigma/dE_T$)

- QCD motivated:
  - Leading parton radiates gluons uniformly distributed azimuthally around jet axis
  - Assume zero-mass particles using calorimeter towers
    - 1 particle per tower
    - Each “particle” will have an energy $k_T$ perpendicular to the jet axis:
    - From energy conservation we expect total energy perpendicular to the jet axis to be zero on average:
      \[
      \sum_{\text{particles}} k_T = 0
      \]
  - Find jet axis that minimizes $k_T$ relative to that axis
  - Use this to define jet 4-vector from calorimeter towers
  - Since calorimeter towers measure total energy, make a basic assumption:
    - Energy of tower $E_i$ is from a single particle with that energy
    - Assume zero mass particle (assume it’s a pion and you will be right >90%!)  
    - Momentum of the particle is then given by
      \[
      \vec{p}_i = E_i \hat{n}_i \quad \text{and} \quad \hat{n}_i \text{ points to tower } i \text{ with energy } E_i
      \]
  - Note: $m_i=0$ does NOT mean $M_{\text{jet}}=0$
    - Mass of jet is determined by opening angle between all contributors
    - Can see this in case of 2 “massless” particles, or energy in only 2 towers:
      \[
      M^2_{12} = 2E_1E_2(1 - \cos \theta_{12}) = 4E_1E_2 \sin^2 \frac{\theta_{12}}{2}
      \]
    - Mass is “generated” by opening angles.
    - A rule of thumb: Zero mass parents of decay have $\theta_{12}=0$ always
Quasi-analytical approach

- Transform each calorimeter tower to frame of jet and minimize $k_T$
  - 2-d Euler rotation (in picture, $\phi = \phi_{\text{jet}}, \theta = \theta_{\text{jet}}, \text{set } \chi = 0$)

$$M\left(\phi_{\text{jet}}, \theta_{\text{jet}}\right) = \begin{pmatrix}
  -\sin \phi_{\text{jet}} & \cos \phi_{\text{jet}} & 0 \\
  -\cos \theta_{\text{jet}} \cos \phi_{\text{jet}} & -\cos \theta_{\text{jet}} \sin \phi_{\text{jet}} & \sin \theta_{\text{jet}} \\
  \sin \theta_{\text{jet}} \cos \phi_{\text{jet}} & \sin \theta_{\text{jet}} \sin \phi_{\text{jet}} & \cos \theta_{\text{jet}}
\end{pmatrix}$$

- Tower in jet momentum frame: $\vec{E}_i' = M\left(\theta_{\text{jet}}, \phi_{\text{jet}}\right) \times \vec{E}_i$ and apply $\sum_{\text{particles}} k_T = 0$

$$E_{xi}' = -E_{xi} \sin \phi_{\text{jet}} + E_{yi} \cos \phi_{\text{jet}}$$
$$E_{yi}' = -E_{xi} \cos \theta_{\text{jet}} \cos \phi_{\text{jet}} - E_{yi} \cos \theta_{\text{jet}} \sin \phi_{\text{jet}} + E_{zi} \sin \theta_{\text{jet}}$$
$$E_{zi}' = E_{xi} \sin \theta_{\text{jet}} \cos \phi_{\text{jet}} + E_{yi} \sin \theta_{\text{jet}} \sin \phi_{\text{jet}} + E_{zi} \cos \theta_{\text{jet}}$$

- Check: for 1 tower, $\phi_{\text{tower}} = \phi_{\text{jet}}$, should get $E_{xi}' = E_{yi}' = 0$ and $E_{zi}' = E_{\text{jet}}$
  - It does, after some algebra…
Minimize $k_T$ to Find Jet Axis

- The equation $\sum_{\text{particles}} \vec{k}_T = 0$ is equivalent to $\sum_i E'_x = \sum_i E'_y = 0$ so...

\[
\sum E'_x = -\sin \phi_{\text{jet}} \sum E_x + \cos \phi_{\text{jet}} \sum E_y = 0 \quad \Rightarrow \quad \tan \phi_{\text{jet}} = \frac{\sum E_y}{\sum E_x}
\]

\[
\sum E'_y = -\cos \theta_{\text{jet}} \left( \cos \phi_{\text{jet}} \sum E_x - \sin \phi_{\text{jet}} \sum E_y \right) + \sin \theta_{\text{jet}} \sum E_z = 0 \quad \Rightarrow \quad \tan \theta_{\text{jet}} = \frac{\sqrt{\left(\sum E_x\right)^2 + \left(\sum E_y\right)^2}}{\sum E_z}
\]

- Momentum of the jet is such that:

\[
\tan \phi_{\text{jet}} = \frac{p_{y,\text{jet}}}{p_{x,\text{jet}}}
\]

\[
p_{x,\text{jet}} = \sum E_x \quad \Rightarrow \quad \tan \theta_{\text{jet}} = \frac{p_{T,\text{jet}}}{p_{z,\text{jet}}}
\]

\[
p_{y,\text{jet}} = \sum E_y \quad \Rightarrow \quad p_{T,\text{jet}} = \sqrt{\left(\sum E_x\right)^2 + \left(\sum E_y\right)^2}
\]

\[
p_{z,\text{jet}} = \sum E_z \quad \Rightarrow \quad p_{z,\text{jet}} = \sum E_z
\]
Jet 4-momentum summary

• Jet Energy:  
\[ E_{\text{jet}} = \sum_{\text{towers}} E_i \]

• Jet Momentum:  
\[ \mathbf{\hat{p}}_{\text{jet}} = \sum_{\text{towers}} E_i \mathbf{\hat{n}}_i \]

• Jet Mass:  
\[ M^2_{\text{jet}} = E^2_{\text{jet}} - p^2_{\text{jet}} \]

• Jet 4-vector:  
\[ p^\mu_{\text{jet}} = (E_{\text{jet}}, \mathbf{\hat{p}}_{\text{jet}}) = \left( \sum_{\text{cells}} E_i, \sum_{\text{cells}} E_i \mathbf{\hat{n}}_i \right) \]

• Jet is an object now! So how do we define $E_T$?
For any object, $E_T$ is well defined:

$$E_{T, \text{jet}} \equiv \sqrt{E_{\text{jet}}^2 - p_{z, \text{jet}}^2} = \sqrt{p_{T, \text{jet}}^2 + m_{\text{jet}}^2}$$

Correct

There are 2 more ways you could imagine using to define $E_T$ of a jet but neither are technically correct:

**Alternative 1**

$$E_{T, \text{jet}} = E_{\text{jet}} \sin \theta_{\text{jet}}$$

**Alternative 2**

$$E_{T, \text{jet}} = \sum_{\text{towers}} E_{T, i}$$

– How do they compare?
– Is there any $E_T$ or $\eta$ dependence?
True $E_T$ vs Alternative 1

- **True:**
  
  $$E_{T,\text{jet}} = \sqrt{p_{T,\text{jet}}^2 + m_{\text{jet}}^2}$$

- **Alternative 1:**
  
  $$E_{T,\text{jet}} = E_{\text{jet}} \sin \theta_{\text{jet}} = \sqrt{p_{\text{jet}}^2 + m_{\text{jet}}^2} \sin \theta_{\text{jet}} = \sqrt{p_{T,\text{jet}}^2 + m_{\text{jet}}^2 \sin^2 \theta_{\text{jet}}}$$

- Define
  
  $$\Delta_1 \equiv \frac{E_{T,\text{jet}} - E_{\text{jet}} \sin \theta_{\text{jet}}}{E_{T,\text{jet}}} = 1 - \frac{\sqrt{p_{T,\text{jet}}^2 + m_{\text{jet}}^2 \sin^2 \theta_{\text{jet}}}}{\sqrt{p_{T,\text{jet}}^2 + m_{\text{jet}}^2}}$$

  which is always $> 0$

  - Expand in powers of $\frac{m_{\text{jet}}^2}{p_{T,\text{jet}}^2}$: $\Delta_1 \rightarrow \frac{m_{\text{jet}}^2 \tanh^2 \eta_{\text{jet}}}{2 p_{T,\text{jet}}^2}$

  - For small $\eta$, $\tanh \eta \rightarrow \eta$ so either way is fine
    - Alternative 1 is the equivalent to true def central jets
      - Agree at few% level for $|\eta| < 0.5$

  - For $\eta \sim 0.5$ or greater....cone dependent
    - Or “mass” dependent....same thing

![Leading jet, $|\eta| > 0.5$](chart.png)
True $E_T$ vs Alternative 2

Alternative 2: \[ E_{T,\text{jet}} = \sum_{\text{towers}} E_{T,i} \]

- **TRUE:**
  \[
  E_{T,\text{jet}}^2 = E_{jet}^2 - p_{z,\text{jet}}^2 = (E_1 + E_2)^2 - (p_{z1} + p_{z2})^2
  \]
  \[
  = E_1^2 + 2E_1E_2 + E_2^2 - p_{z1}^2 + 2p_{z1}p_{z2} + p_1^2
  \]
  \[
  = E_{T1}^2 + E_{T2}^2 + 2E_1E_2(1 - \cos \theta_1 \cos \theta_2)
  \]

- Alternative 2:
  \[
  (E_{T1} + E_{T2})^2 = E_{T1}^2 + E_{T2}^2 + 2E_{T1}E_{T2}
  \]
  \[
  = E_{T1}^2 + E_{T2}^2 + 2E_1E_2 \sin \theta_1 \sin \theta_2
  \]

- Take difference:
  \[
  E_{T,\text{jet}}^2 - (E_{T1} + E_{T2})^2 = 2E_1E_2(1 - \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2)
  \]
  \[
  = 2E_1E_2(1 - \cos \delta \theta) = E_1E_2 \sin^2 \delta \theta / 2
  \]
  Always $> 0!$

- So this method also underestimates “true” $E_T$
  - But not as much as Alternative 1
Jet Shape

• Jets are defined by $\sum_{\text{particles}} \vec{k}_{T,i} = 0$ but the “shape” is determined by

$$\sum_{\text{particles}} k_{T,i}^2 = \sum_{\text{particles}} E_{x,i}^2 + E_{y,i}^2 \geq 0$$

• From Euler:

$$E_{xi}' = -E_{xi} \sin \phi_{jet} + E_{yi} \cos \phi_{jet} = E_{Ti} \sin \delta \phi_i$$

$$E_{yi}' = -E_{xi} \cos \theta_{jet} \cos \phi_{jet} - E_{yi} \cos \theta_{jet} \sin \phi_{jet} + E_{zi} \sin \theta_{jet}$$

$$= -E_{Ti} \cos \delta \phi_i \cos \theta_{jet} + E_{zi} \sin \theta_{jet}$$

$$\delta \phi \equiv \phi_i - \phi_{jet}$$

$$\delta \theta \equiv \theta_i - \theta_{jet}$$

• Now form $\sum_{\text{particles}} k_{T,i}^2$ for those towers close to the jet axis: $\delta \theta \rightarrow 0$ and $\delta \phi \rightarrow 0$

$$E_{xi}' \rightarrow E_{Ti} \delta \phi_i$$

$$E_{yi}' \rightarrow -E_{Ti} \cos \theta_{jet} + E_{zi} \sin \theta_{jet} = -E_i \sin \theta_i \cos \theta_{jet} + E_i \cos \theta_i \sin \theta_{jet} = E_i \sin \delta \theta_i \sim E_i \delta \theta_i$$

• From $\tanh \eta = \cos \theta$ we get $d\theta = -\sin \theta d\eta$ which means

$$E_{xi}' \rightarrow E_{Ti} \delta \phi_i$$

$$E_{yi}' \rightarrow E_i \delta \theta_i = -E_i \sin \theta_i \delta \eta_i \rightarrow -E_{Ti} \delta \eta_i$$

So…

$$k_{T,i}^2 = E_{xi}'^2 + E_{yi}'^2 = E_{T,i}^2 \left( \delta \phi_i^2 + \delta \eta_i^2 \right)$$

and…

$$\sum_{\text{particles}} k_{T,i}^2 = \sum_{\text{particles}} E_{x,i}^2 + E_{y,i}^2 = \sum_{\text{particles}} E_{T,i}^2 \left( \delta \phi_i^2 + \delta \eta_i^2 \right)$$
Jet Shape – $E_T$ Weighted

- Define $\delta R_i^2 \equiv \delta \phi_i^2 + \delta \eta_i^2$ and $\delta R_i = \sqrt{\delta R_i^2} = \sqrt{\delta \phi_i^2 + \delta \eta_i^2}$

  - This gives: $\sum_{\text{particles}} k_{T,i}^2 = \sum_{\text{particles}} E_{T,i}^2 \delta R_i^2$ and equivalently, $k_{T,i} = E_{T,i} \delta R_i$

- Momentum of each “cell” perpendicular to jet momentum is from
  - $E_{\|}$ of particle in the detector, and
  - Distance from jet in $\eta \phi$ plane

- This also suggests jet shape should be roughly circular in $\eta \phi$ plane
  - Providing above approximations are indicative overall….

- Shape defined:
  - Use energy weighting to calculate true $2^{\text{nd}}$ moment in $\eta \phi$ plane

\[
\sigma_R^2 \equiv \frac{\sum_{\text{particles}} k_{T,i}^2}{\sum_{\text{particles}} E_{T,i}^2} = \frac{\sum_{\text{particles}} E_{T,i}^2 \delta R_i^2}{\sum_{\text{particles}} E_{T,i}^2} = \sigma_\eta + \sigma_\phi \\
\text{with } \sigma_\eta \equiv \frac{\sum_{\text{particles}} E_{T,i}^2 \delta \eta_i^2}{\sum_{\text{particles}} E_{T,i}^2}, \quad \sigma_\phi \equiv \frac{\sum_{\text{particles}} E_{T,i}^2 \delta \phi_i^2}{\sum_{\text{particles}} E_{T,i}^2}
\]
Jet Shape – E_T Weighted (cont)

- Use sample of “unmerged” jets

- Plot

  \[ \sigma_R = \sqrt{\frac{\sum_{\text{particles}} E_{x,i}^2 + E_{y,i}^2}{\sum_{\text{particles}} E_{T,i}^2}} \]

  - Shape depends on cone parameter
  - Mean and widths scale linearly with cone parameter

“Small angle” approximation pretty good

- For Cone=0.7, distribution in \( \sigma_R \) has:
  - Mean ± Width = .25 ± .05
  - 99% of jets have \( \sigma_R < 0.4 \)
Jet Mass
Jet Samples

- DZero Run 1
- All pathologies eliminated (Main Ring, Hot Cells, etc.)
- $|Z_{vtx}| < 60$cm
- No $\tau$, $e$, or $\gamma$ candidates in event
  - Checked $\eta\phi$ coords of $\tau e \gamma$ vs. jet list
  - Cut on cone size for jets
    - .025, .040, .060 for jets from cone cutoff 0.3, 0.5, 0.7 respectively
- "UNMERGED" Sample:
  - RECO events had 2 and only 2 jets for cones .3, .5, and .7
  - Bias against merged jets but they can still be there
    - e.g. if merging for all cones
- "MERGED" Sample:
  - Jet algorithm reports merging
Jet Mass

- Jet is a physics object, so mass is calculated using:
  - Either one…

\[
M_{jet}^2 = E_{jet}^2 - p_{jet}^2 = E_{T,jet}^2 - p_{T,jet}^2
\]

- Note: there is no such thing as “transverse mass” for a jet
  - Transverse mass is only defined for pairs (or more) of 4-vectors…

- For large \(E_{T,jet}\) we can see what happens by writing

\[
M_{jet}^2 = E_{T,jet}^2 - p_{T,jet}^2 = \left( E_{T,jet} + p_{T,jet} \right) \left( E_{T,jet} - p_{T,jet} \right)
\]

  - And take limit as jet narrows \(\delta \eta_i \to 0\) and \(\delta \phi_i \to 0\) and expand \(E_T\) and \(p_T\)

\[
p_{T,jet} \to \sum E_{T,i} \left( 1 - \frac{\delta \phi_i^2}{2} \right) \quad E_{T,jet} \to \sum E_{T,i} \left( 1 + \frac{\delta \eta_i^2}{2} \right)
\]

  - This gives

\[
E_{T,jet} - p_{T,jet} = \frac{1}{2} \sum E_{T,i} \left( \delta \eta_i^2 + \delta \phi_i^2 \right) \quad E_{T,jet} + p_{T,jet} = \frac{1}{2} \sum E_{T,i} \left( 4 + \delta \eta_i^2 - \delta \phi_i^2 \right) \approx 2 \sum E_{T,i}
\]

Jet mass is related to jet shape!!! (in the thin jet, high energy limit)

so….

\[
M_{jet}^2 = \sum E_{Ti} \sum E_{Ti} \left( \delta \eta_i^2 + \delta \phi_i^2 \right) \rightarrow M_{jet} \equiv E_{T,jet} \sigma_R \text{ using } E_{T,jet} \equiv \sum E_{T,i} \text{ particles}
\]
Jet Mass (cont)

- Jet Mass for unmerged sample

![Graph showing jet mass distribution for different cone sizes.]

- How good is “thin jet” approximation?

![Graph showing comparison of jet mass distributions with and without thin jet approximation.]

Low-side tail is due to lower $E_T$ jets for smaller cones
(this sample has 2 and only 2 jets for all cones)
Jet Merging

- Does jet merging matter for physics?
  - For some inclusive QCD studies, it doesn’t matter
  - For invariant mass calculations from e.g. top→Wb, it will smear out mass distribution if merging two “tree-level” jets that happen to be close

- Study $\sigma_R$…see clear correlation between $\sigma_R$ and whether jet is merged or not
  - Can this be used to construct some kind of likelihood?

"Unmerged", Jet Algorithm reports merging, all cone sizes

```
10-Dec-2008  D. Baden, U. Geneve
```
Merging Likelihood

- **Crude attempt at a likelihood**
  - Can see that for this (biased) sample, can use this to pick out “unmerged” jets based on shape
  - Might be useful in Higgs search for $H \rightarrow bb$ jet invariant mass?

<table>
<thead>
<tr>
<th>Jet cone parameter</th>
<th>Equal likelihood to be merged and unmerged</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.155</td>
</tr>
<tr>
<td>0.5</td>
<td>0.244</td>
</tr>
<tr>
<td>0.7</td>
<td>0.292</td>
</tr>
</tbody>
</table>
Merged Shape

- Width in $\eta\phi$ \[ \sigma_R^2 = \sigma_{\eta\eta} + \sigma_{\phi\phi} \] “assumes” circular

- Large deviations due to merging?

- Define \[ \delta_{\eta\phi} \equiv \frac{\sigma_{\eta\eta} - \sigma_{\phi\phi}}{\sigma_{\eta\eta} + \sigma_{\phi\phi}} \] should be independent of cone size

- Clear broadening seen – “cigar”-shaped jets, maybe study…

\[ \sigma_{\eta\eta} = \sum_{\text{particles}} \frac{E_{T,i}^2 \delta_{\phi} \delta_{\eta} \sum_{\text{particles}} E_{T,i}^2}{\sum_{\text{particles}}} \]