Homework 5

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1.a Thermal equilibrium means that the temperature T at the box $r = R$ is the blue-shifted Hawking temperature $T_H = \frac{1}{8\pi M}$, where M is the mass of the hole. This equilibrium condition gives a curve of solutions

$$T(M) \quad = \quad \frac{1}{\sqrt{1-\frac{2M}{R}}} T_H = \frac{R}{8\pi M\sqrt{R-2M}} \ . \tag{1}$$

Note that $T(M)$ is positive on $M \in (0, R/2)$ and diverges at the end points of this interval. In particular, it has a minimum at $\partial T/\partial M = 0$ which is equivalent to a maximum of the denominator in (1)

$$0 = \frac{\partial}{\partial M}\left(M\sqrt{R-2M}\right) \ \Leftrightarrow \ M = M_* := \frac{R}{3} \ . \tag{2}$$

The critical value $T_* := T(M_*)$ at which this happens is $T_* = \frac{3\sqrt{3}}{8\pi R}$. Since $T \to +\infty$ at $\{0, R/2\}$ and there is only one solution to (2), we conclude that $T(M)$ is convex and that T_* is a minimum. It follows that for $T > T_*$ there are exactly two solutions $M_\pm$ to (1).

1.b.i This follows immediately from the previous discussion: T_* is a global minimum on $(0, R/2)$ and it is the unique extremum. Hence, for $M \in (0, R/3)$ the slope $\chi^{-1} := \partial T/\partial M$ is negative while for $M \in (R/3, R/2)$, it is positive. If χ^{-1} is negative (positive), its inverse (the specific heat χ) is negative (positive).

1.b.ii This also follows immediately from part **a**: The stable holes live on $M \in (R/3, R/2)$. In particular $M > R/3$.

2.a The total entropy of a universe with a black hole of mass M and radiation of energy E_γ can be written as $S_{\text{tot}} = S_{\text{bh}}(M) + S_\gamma(E_\gamma)$. This entropy has a stationary point

$$0 \quad = \quad \delta S_{\text{tot}} = \frac{\partial S_{\text{bh}}}{\partial M}\delta M + \frac{\partial S_\gamma}{\partial E_\gamma}\delta E_\gamma \ . \tag{3}$$

The total energy $E_{\rm tot} = M + E_\gamma$ is taken to be fixed in the micro-canonical ensemble. In other words $0 = \delta M + \delta E_\gamma$ and (3) reduces to

$$0 \;=\; \frac{\partial S_{\rm bh}}{\partial M} - \frac{\partial S_\gamma}{\partial E_\gamma} \;. \tag{4}$$

The stationary point is a local maximum if $\delta^2 S < 0$, *id est*

$$\begin{aligned} 0 \;&>\; \frac{\partial^2 S_{\rm bh}}{\partial M^2}\delta M^2 + \frac{\partial^2 S_\gamma}{\partial E_\gamma^2}\delta E_\gamma^2 \\ &=\; \left[\frac{\partial^2 S_{\rm bh}}{\partial M^2} + \frac{\partial^2 S_\gamma}{\partial E_\gamma^2}\right]\delta M^2 \;. \end{aligned} \tag{5}$$

Now $S_{\rm bh} = 4\pi M^2$. On the other hand, the energy and entropy density of radiation as a function of temperature go like $u \propto T^4$ and $s \propto T^3$, respectively. Let V denote the volume of the box and define constants b and b' such that we have for the total energy and entropy

$$\begin{aligned} E_\gamma \;&=\; bT^4V \;, \\ S_\gamma \;&=\; b'T^3V \;. \end{aligned} \tag{6}$$

(From $dE = TdS$ we can infer $b' = (4/3)b$, but this is not needed below.) Solving the first for T and plugging into the second gives $S_\gamma = cE_\gamma^{3/4}V^{1/4}$ for some c. Plugging this into (4) and (5) respectively gives

$$8\pi M \;=\; \frac{3c}{4}\left(\frac{V}{E_\gamma}\right)^{\frac{1}{4}} \;, \tag{7}$$

and

$$8\pi \;<\; \frac{3c}{4}\left(\frac{V}{E_\gamma}\right)^{\frac{1}{4}}\frac{1}{4E_\gamma} \;. \tag{8}$$

Dividing (7) by (8) gives $E_\gamma < M/4$.

2.b This follows immediately from (7) which we will write as $V = aE_\gamma M^4$ since, E_γ and M, being only part of the total energy of the system E, are both less than E. Therefore $V < aE^5$. However, the maximum value of $E_\gamma M^4$ at fixed E occurs when $E_\gamma = M/4$, i.e. when $E_\gamma = E/5$ and $M = 4E/5$. Hence we have the stronger inequality $V < 4^4 5^{-5} aE_\gamma M^4$.

2.c We will call the case of pure radiation at entropy E case 1 and the case with the black hole case 2. S_1 is the entropy of radiation of energy E at volume V; $S_1 = S_\gamma(E,V)$. For case 2 we have an entropy $S_2 = S_{\rm bh}(M) + S_\gamma(E_\gamma, V)$. Temporarily parameterizing $S_{\rm bh} = \alpha M^2$ and $S_\gamma = \beta E_\gamma^{3/4}$ and using (4) we find that

$$S_{\rm bh} \;=\; \frac{3M}{8E_\gamma}S_\gamma(E_\gamma, V) \;. \tag{9}$$

Substituting $M = 4E_\gamma$ we find $S_2 = (5/2)S_\gamma(E_\gamma, V)$. Finally, we substitute $E_\gamma = E/5$ to get for the ratio

$$\frac{S_2}{S_1} = \frac{5}{2}\cdot\frac{S_\gamma(E/5,V)}{S_\gamma(E,V)} = \frac{5}{2}\cdot\frac{\beta(E/5)^{3/4}}{\beta E^{3/4}} = \frac{5^{1/4}}{2} \approx \frac{3}{4}\ . \tag{10}$$

3.a The surface gravity is the horizon value of $\kappa = |\nabla|\xi||$, with $\xi = \partial/\partial t$ in the Schwarzschild-like co-ordinates. Hence $|\xi|^2 = g_{\alpha\beta}\xi^\alpha\xi^\beta = g_{tt} = f(r)$. Since $|\cdot|$ is a scalar, we get simply $\kappa = \sqrt{-g^{\alpha\beta}\partial_\alpha|\xi|\partial_\beta|\xi|}$. Furthermore, since $|\xi| = \sqrt{f(r)}$ is a function only of the radial co-ordinate, $\kappa = \sqrt{-g^{rr}(\partial_r\sqrt{f})^2} = \sqrt{f}\partial_r\sqrt{f} = \frac{1}{2}f'$. Restricting to $r = r_+$ finally gives

$$\kappa = \frac{r_+}{R^2} + \frac{r_0^2}{r_+^3} = \frac{2r_+}{R^2} + \frac{1}{r_+}\ , \tag{11}$$

where in the second equality we have solved $f(r_+) = 0$ for $r_0^2 = r_+^2 + r_+^4/R^2$ and made the substitution.

3.b The temperature must be evaluated by Wick rotating the AdS-Schwarzschild solution and finding the correct periodic identification. This proceeds the same way as it did for the Schwarzschild black hole with the result that $T_H = \kappa/2\pi$. However, to determine the minimum horizon radius for which the black hole has positive specific heat, it suffices to know simply that this calculation results in $T \propto \kappa$. For the energy, we should compute the charge associated with the Killing field ξ. This turns out to be proportional to r_0^2, the coefficient of the $1/r^2$ term in the metric, which (see 3.a) increases monotonically with r_+. Thus, the sign of the specific heat is the same as the sign of

$$\partial\kappa/\partial r_+ = \frac{2}{R^2} - \frac{1}{r_+}\ . \tag{12}$$

Hence, the specific heat is positive for $r_+ > R/\sqrt{2}$. The temperature at $r_+ = R/\sqrt{2}$ is given by $T_H = \kappa/2\pi = \sqrt{2}/\pi R$.