HW#4 —Phys776—Spring2005 Prof. Ted Jacobson
Due Thursday, March 17, before class. Room 4115, (301)405-6020
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1. The past light cone of a point in Minkowski spacetime is a null hypersurface whose generators form a null geodesic congruence. Consider a point on the cone where the radius of a spherical cross section is r_0. The expansion θ_0 at this point is negative, hence according to the focusing theorem θ will diverge to minus infinity in a finite affine parameter.

   (a) Show that the radial coordinate r is an affine parameter along the null generators of the cone.

   (b) Using r as the affine parameter, what is the value of θ_0 in terms of r_0?

   (c) At what value of the affine parameter range Δr does the focusing theorem predict θ will diverge? Compare your answer to the value of Δr that you infer directly from the definition of the cone, and show that the two answers agree.

2. Consider any static spherically symmetric metric in Schwarzschild-like coordinates,

$$ds^2 = f(r)\,dt^2 - g(r)\,dr^2 - r^2\,d\Omega^2. \tag{1}$$

   (a) Show that the ingoing and outgoing radial light rays are always geodesics.

   (b) Show that r is an affine parameter for the radial null geodesics if and only if the product $f(r)g(r)$ is constant.

   (c) Use the Raychaudhuri equation to show that r is an affine parameter on the radial outgoing or ingoing null geodesics if and only if $R_{ab}k^a k^b = 0$, where k^a is the tangent vector to these geodesics.

   (d) The above reasoning shows that in a vacuum solution $g(r) \propto 1/f(r)$. Use the Einstein equation to show that this conclusion still holds when (*i*) there is a cosmological constant term and (*ii*) there is a radial electric or magnetic field present, but (*iii*) not when there is a non-constant scalar field. [For the electromagnetic and massless scalar stress tensors see, e.g., eqns. (3.3) and (4.56) in the lecture notes by Townsend at the course web page, or see http://arxiv.org/abs/hep-th/0411105 which treats dilaton and Maxwell fields in the presence of a cosmological constant.]

3. In class we derived the "dynamical process version" of the first law for small perturbations of a black hole horizon. For perturbations that are very nearly stationary in time there is a more restrictive "adiabatic version" of the first law. This is explained in equations (11) and (12) of my paper "Horizon Entropy", listed at the course web page.

   (a) Derive equation (11) for the expansion with respect to Killing time from the usual Raychaudhuri equation. (Note that strictly speaking there is no "Killing time" in the perturbed solution, since it has no Killing symmetry. What is meant here is to derive (11) with κ defined as below Eqn. (11), by $\lambda_{,vv}/\lambda_{,v} = \kappa$. In the slowly evolving spacetime, the 'adiabatic Killing time' v can be defined by integrating this equation, with κ equal to the (approximately defined) 'instantaneous surface gravity'.)

(b) Exhibit the steps that establish the claim that eqn (4) follows from (5) and (12). (Be careful about the distinction between θ and $\hat{\theta}$.)

4. Consider a process in which a thin spherical shell of mass M collapses to form a black hole, and then after a Killing time Δv in this black hole spacetime a second shell of mass ΔM collapses, forming a black hole of mass $M + \Delta M$.

   (a) Draw a spacetime diagram of the process just described. Show both shells, the horizon if $\Delta M = 0$, and the horizon if $\Delta M \neq 0$.

   (b) The horizon has three distinct sections: (*i*) the flat null cone inside the first shell, (*ii*) an expanding null hypersurface in the Schwarzschild metric of mass M between the shells, and (*iii*) a null hypersurface of constant cross sectional area outside the second shell. Find the the radius r (i.e. $\sqrt{\text{area}/4\pi}$) of section (*ii*) as a function of the advanced time coordinate v. Show that the true event horizon approaches the would-be event horizon for the mass M exponentially quickly toward the past as $\sim \exp(\kappa v)$, where $\kappa = 1/4M$ is the surface gravity.

   The advanced time coordinate is defined by $v = t + r_*$, where r_* is the "tortoise coordinate" defined by $dr_* = dr/(1 - 1/r)$, or $r_* = r + \ln(r - 1)$, in units with $2M = 1$. Note that $dv = 0$ on an ingoing radial light ray. In the (*Eddington-Finkelstein*) coordinate system (v, r, θ, ϕ), the line element takes the form $ds^2 = (1 - 1/r)dv^2 - 2dvdr - r^2 d\Omega^2$, and the time translation Killing vector is ∂_v. Note that, unlike t, v is regular at the horizon.