Homework 3

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1.a The rate of change of a vector v^a flowing along the integral curves of u^a is given by $\dot{v}^a = u^b\nabla_b v^a = v^b\nabla_b u^a = B^a{}_b v^b = \frac{1}{3}\theta v^a + \sigma^a{}_b v^b + \omega^a{}_b v^b$ for $B^a{}_b := \nabla_b u^a$ where we have used the fact that the vector fields u and v commute. It follows from this that the length of v changes by $\frac{d}{d\lambda}|v| = \frac{1}{|v|}(\frac{1}{3}\theta v^2 + \sigma_{ab}v^a v^b) = \frac{1}{3}\theta|v| + \sigma_{ab}v^a v^b/|v|$. Now suppose that v^a is one of a set of three spacial vector fields spanning the $t =$ constant sections in a Robertson-Walker spacetime, i. e. the co-moving frame. By isotropy, the shear for the flow of this vector must vanish, for if it did not, a round sphere at $t = t_0$ would evolve into a squashed sphere for $t > t_0$ which is certainly not invariant under rotations about its center. Hence $\frac{d}{d\lambda}|v| = \frac{1}{3}\theta|v|$. Now the metric for a Robertson-Walker spacetime can be put in the form

$$ds^2 \;=\; dt^2 - a^2(t)d\Sigma^2 \;, \tag{1}$$

where $d\Sigma^2$ is the volume element of a unit (pseudo)sphere or Euclidean 3-space. In other words, all lengths on Σ are determined by $a(t)$. In particular, the spacial sections are expanding uniformly in all directions at the rate

$$\dot{a} = \frac{1}{3}\theta a \tag{2}$$

as we have just seen.[1] Consequently, $\ddot{a} = \frac{1}{3}(\dot{\theta}a + \frac{1}{3}\theta^2 a)$. Then, by the shearless, twistless, Raychaudhuri equation, $\ddot{a}/a = -\frac{1}{3}R_{ab}u^a u^b$. The "trace-reverse" of the standard Einstein equation, $R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab}$, is $R_{ab} = 8\pi(T_{ab} - \frac{1}{2}g_{ab}T)$. Futhermore, the stress-energy of a perfectly homogeneous and isotropic co-moving cosmological fluid is given in the coordinate system of (1) as $(T_{\alpha\beta}) = \mathrm{diag}(\rho, -p, -p, -p)$, where ρ is the energy density and p is the pressure. Plugging this in gives

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{8\pi}{3}\left(\rho - \frac{1}{2}(\rho - 3p)\right) \\ &= -\frac{4\pi}{3}(\rho + 3p)\;, \end{aligned} \tag{3}$$

[1] It is worth emphasizing here that the expansion represented by θ is that of the co-moving frame or "cosmological fluid" (also known as the (in)famous "æther").

as desired.

1.b If the test particles are initially at rest with respect to each other and not rotating about their center of mass, then for a vector v denoting the displacement between two distinct particles, $\dot{v}^a = 0$. Since this must hold for *any* such (spacial) vector this implies $B^a{}_b = 0$ (see **1.a**) which, in turn, implies $\theta = \sigma = \omega = 0$. Finally, since its center is at rest w. r. t. the cosmological fluid, then in the coordinate system of (1) $u = \partial_t$. In this situation the Raychaudhuri equation simplifies to $\dot{\theta} = -R_{tt} = -4\pi(\rho + 3p)$. Hence, for $\rho + 3p < 0$, $\dot{\theta} < 0$ and the ball is contracting.

This is not inconsistent with the expansion of the universe computed in **1.a**. As emphasized in the previous footnote, the expansion computed in that case was that of the cosmological fluid while in this case the same symbol denotes the expansion of a ball of test particles. In the latter case, for example, we took the initial condition on the Raychaudhuri equation to be $\theta = 0$. This is simply not a consistent initial condition for the cosmological fluid as it would imply that $\dot{a} = 0$ by (2) and therefore $a = 0$ for all time.

2. *Nota bene*: My convention for anti-symmetrization differs from Ted's by factors of $p!$. For example, in my notation $\epsilon_{[abcd]} = 4!\epsilon_{abcd}$ and $\nabla_{[a}\nabla_{b]} = [\nabla_a, \nabla_b]$ as opposed to Ted's notational convention which would give $1 \cdot \epsilon_{abcd}$ and $\nabla_{[a}\nabla_{b]} = \frac{1}{2}[\nabla_a, \nabla_b]$ respectively.

2.a $\nabla_{[a}V_{b]} = \nabla_{[a}(f\nabla_{b]}S) = \nabla_{[a}f\nabla_{b]}S + f[\nabla_a, \nabla_b]S$. The last term vanishes. Thus, multiplying and dividing by f, we obtain $\nabla_{[a}V_{b]} = (\nabla_{[a}f)f^{-1}(f\nabla_{b]}S) = V_{[a}W_{b]}$ for $W_a := -\nabla_a \log f$.

2.b Clearly $\nabla_{[a}V_{b]} = V_{[a}W_{b]}$ implies $V_{[a}\nabla_b V_{c]} = 0$ since multiplying the first by V_c and antisymmetrizing forces us to antisymmetrize on $V_a V_b$. Now, as in the hint, take any X^a such that $X \cdot V \neq 0$. Then

$$\begin{aligned} 0 &= X^c V_{[a}\nabla_b V_{c]} \\ &= X^c \left(\nabla_{[b}V_{c]} + V_b\nabla_{[c}V_{a]} + V_c\nabla_{[a}V_{b]}\right) \\ &= X^c \left(V_a\nabla_{[b}V_{c]} - V_b\nabla_{[a}V_{c]}\right) + (X \cdot V)\nabla_{[a}V_{b]} \\ &= \left[V_a(X^c\nabla_{[b}V_{c]}) - V_b(X^c\nabla_{[a}V_{c]})\right] + (X \cdot V)\nabla_{[a}V_{b]} \ . \end{aligned} \tag{4}$$

Now define $W_a := -(X \cdot V)^{-1}X^c\nabla_{[a}V_{c]}$. Then the equation above implies that $\nabla_{[a}V_{b]} = V_{[a}W_{b]}$, which is the desired relation.

2.c ∂_t is obviously orthogonal to the $dt = 0$ hypersurfaces but it is instructive to see from the relation derived in **2.b** why this is so. The Minkowski metric in spherical coordinates is

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{5}$$

Let $\alpha = g(\partial_t, \cdot)$ be the 1-form (co-vector) gotten from the vector $(\partial_t)^a$ by lowering the index with the metric, in coordinates $\alpha_\gamma = g_{t\gamma} = \delta_{t\gamma}$. Since this co-vector has only one non-vanishing component[2] $\alpha_{[a}\nabla_b\alpha_{c]} = \alpha_{[a}\partial_b\alpha_{c]} \equiv 0$ so that the vector ∂_t is hypersurface

[2]Notice that for any co-vector V_a, $\nabla_{[a}V_{b]} = \partial_{[a}V_{b]} - \Gamma^c_{[ab]}V_c = \partial_{[a}V_{b]}$ in the absence of torsion, i. e. the exterior derivative is covariant.

orthogonal by the results of **2.a** and **2.b**. Notice that this implies that *any* coordinate vector[3] is hypersurface orthogonal if the metric has no off-diagonal terms in that coordinate– in particular, so is ∂_ϕ. However, let us now consider the vector field generated by $\partial_t + \Omega\partial_\phi$ and define the 1-form $\nu := g(\partial_t + \Omega\partial_\phi, \cdot)$ by lowering the index. In the coordinates of (5) this gives $(\nu_\alpha) = (1, 0, 0, -\Omega r^2 \sin^2\theta)$. Now consider again the equation $\nu_{[\alpha}\nabla_\beta\nu_{\gamma]} = \nu_{[\alpha}\partial_\beta\nu_{\gamma]}$ but for the coordinate directions $\alpha = t$ and $\gamma = \phi$ leaving β free for the moment. Then

$$\begin{aligned}
\nu_{[\alpha}\partial_\beta\nu_{\gamma]} &\to \nu_{[t}\partial_\beta\nu_{\phi]} \\
&= \nu_t\partial_\beta\nu_\phi + \nu_\beta\partial_\phi\nu_t + \nu_\phi\partial_t\nu_\beta \\
&- \nu_t\partial_\phi\nu_\beta - \nu_\beta\partial_t\nu_\phi - \nu_\phi\partial_\beta\nu_t \\
&= \nu_t\partial_\beta\nu_\phi \ ,
\end{aligned} \tag{6}$$

where all the other terms vanish either because ν_t is constant or $g_{\alpha\beta}$ is independent of t and ϕ. By taking $\beta = r$ or $\beta = \theta$ we can see that the equation for hypersurface orthogonality is not satisfied.

2.d Recall that in the case of the Kerr metric in Boyer-Lindquist coordinates g_{tt}, $g_{t\phi}$ and $g_{\phi\phi}$ are all non-vanishing. Therefore the co-vectors $\alpha = g(\partial_t, \cdot)$ and $\beta = g(\partial_\phi, \cdot)$ are given by

$$\begin{aligned}
(\alpha_\alpha) &= (g_{tt}, 0, 0, g_{t\phi}) \\
(\beta_\alpha) &= (g_{t\phi}, 0, 0, g_{\phi\phi}) \ .
\end{aligned} \tag{7}$$

Both co-vectors are of the form $\gamma = (f, 0, 0, g)$ with f and g independent of t and ϕ so considering again the equation (6) we find that

$$\gamma_{[t}\partial_\beta\gamma_{\phi]} = f\partial_\beta g - g\partial_\beta f \ . \tag{8}$$

Given the explicit forms of f and g for the two cases above, we see that a miracle is required such that this expression vanish for all $\beta = r, \theta$. The miracle does not happen.

2.e Since χ is hypersurface orthogonal on the horizon, we can apply **2.a** there to write the twist $\omega_{ab} = \nabla_{[a}\chi_{b]} = \chi_{[a}v_{b]}$ for some v. Notice that since χ is a Killing vector, the geodesic equation $\chi^a\nabla_a\chi_b = \kappa\chi_b$ reduces to $\kappa\chi_b = \frac{1}{2}\chi^a\nabla_{[a}\chi_{b]} = \frac{1}{2}\chi^a\chi_{[a}v_{b]} = -\frac{1}{2}(\chi\cdot v)\chi_b$ since χ is a null vector. Consequently

$$\kappa = -\frac{1}{2}(\chi \cdot v) \ . \tag{9}$$

Now square this to obtain

$$\begin{aligned}
\kappa^2 &= \frac{1}{4}(\chi_a v_b)(\chi^b v^a) \\
&= \frac{1}{4}(\chi_a v_b)(\chi^{[b}v^{a]}) \\
&= -\frac{1}{2}\left(\frac{1}{2}\chi_{[a}v_{b]}\right)\left(\frac{1}{2}\chi^{[a}v^{b]}\right)
\end{aligned}$$

[3] A vector v is a *coordinate vector* if there exists a coordinte system $\{x^\alpha\}$ in which v can be written as $v^a = (\partial/\partial x^\alpha)^a$.

$$
\begin{aligned}
&= -\frac{1}{2}\left(\frac{1}{2}\nabla_{[a}\chi_{b]}\right)\left(\frac{1}{2}\nabla^{[a}\chi^{b]}\right) \\
&= -\frac{1}{2}(\nabla_a\chi_b)(\nabla^a\chi^b) \ , \qquad (10)
\end{aligned}
$$

where in the second line we have used the nullity of χ and in the last we have used the Killinginity.

2.f We take u^a to be a hypersurface orthogonal

$$u_{[a}\nabla_b u_{c]} \ = \ 0 \ , \qquad (11)$$

timelike or spacelike

$$u^2 \ \neq \ 0 \ , \qquad (12)$$

affinely parameterized geodesic vector field

$$u^a\nabla_a u^b \ = \ 0 \ , \qquad (13)$$

and simply compute

$$
\begin{aligned}
0 &\overset{(11)}{=} u^a\left(u_a\omega_{bc} + \left[u_b\nabla_{[c}u_{a]} - (b\leftrightarrow c)\right]\right) \\
&= u^2\omega_{bc} + \left[\tfrac{1}{2}u_b\nabla_c u^2 - u_b u^a\nabla_a u_c - (b\leftrightarrow c)\right] \\
&\overset{(13)}{=} u^2\omega_{bc} + \tfrac{1}{2}u_{[b}\nabla_{c]}u^2 \ . \qquad (14)
\end{aligned}
$$

Since we can choose an affine parameterization such that u^2 is constant over the entire congruence, the second term can be made to vanish. Equation (12) then implies that the twist vanishes.

2.g From the condition that k^a generates an affinely parameterized geodesic, $0 = k^b\nabla_b k_a$. Subtracting the condition that k is null, $0 = \frac{1}{2}\nabla_a k^2 = k^b\nabla_a k_b$, we find that $0 = k^b\nabla_b k_a - k^b\nabla_a k_b = k^b\omega_{ba}$. If k^a is hypersurface orthogonal then by **2.a** $\omega_{ab} = \nabla_{[a}v_{b]} = k_{[a}v_{b]}$ for some v_b and we find that $0 = k^b(k_a v_b - k_b v_a) = k_a(k\cdot v)$. Since this must hold for all null k, we find that v must be orthogonal to k. Given this form of ω it follows that $\omega_{ab}\omega^{ab} = 2(k_a v_b - k_b v_a)k^a v^b = 2k^2v^2 - 2(k\cdot v)^2 = 0$.