Homework 2

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March 2, 2005

1. We work in a coordinate system adapted to ξ and compute

$$\nabla_\alpha \xi_\beta \;=\; (\nabla_\alpha \xi^\gamma) g_{\gamma\beta} = (\xi^\gamma_{,\alpha} + \Gamma^\gamma_{\delta\alpha}\xi^\delta) g_{\gamma\beta} \; . \tag{1}$$

Since the coordinates $\{x^\alpha\}$ are adapted to ξ, the components $\xi^\gamma = \delta^\gamma_{\hat\alpha}$ for some coordinate direction $\hat\alpha$. In particular, $\xi^\gamma_{,\alpha} \equiv 0$ and

$$\begin{aligned} 2\nabla_\alpha \xi_\beta &= g^{\gamma\epsilon}(g_{\epsilon\delta,\alpha} + g_{\epsilon\alpha,\delta} - g_{\delta\alpha,\epsilon}) g_{\gamma\beta}\xi^\delta \\ &= (2g_{\delta[\beta,\alpha]} + g_{\alpha\beta,\delta})\xi^\delta \; , \end{aligned} \tag{2}$$

It follows that

$$\nabla_{(\alpha}\xi_{\beta)} \;\propto\; \frac{\partial g_{\alpha\beta}}{\partial x^{\hat\delta}} \xi^{\hat\delta} \; . \tag{3}$$

Since $g_{\alpha\beta}$ does not depend on $x^{\hat\alpha}$ this vanishes. Since 0 is a tensor, we find in general that the Killing condition is $\nabla_{(a}\xi_{b)} = 0$.

2.a Orthogonality of χ and ∂_ϕ at the horizon, H, gives

$$0 \;=\; \chi \cdot \partial_\phi|_H = \frac{2Mr_+ a \sin^2\theta}{\Sigma|} - \Omega_H \frac{A|\sin^2\theta}{\Sigma|} \;\Rightarrow\; \Omega_H = \frac{2Mr_+ a}{A|} \; . \tag{4}$$

Now, at the horizon $\Delta| = 0$ which implies $2Mr_+/(r_+^2 + a^2) = 1$ and $A| = (r_+^2 + a^2)^2$. This gives

$$\Omega_H \;=\; \frac{a}{r_+^2 + a^2} = \frac{a}{2Mr_+} \; . \tag{5}$$

2.b Let $\tilde\kappa := \sqrt{-g^{ab}\nabla_a|\chi|\nabla_b|\chi|}$ so that $\kappa = \lim_{r\to r_+}\tilde\kappa$. Since $|\chi|$ is a scalar, $\tilde\kappa = \sqrt{-g^{\alpha\beta}\partial_\alpha|\chi|\partial_\beta|\chi|}$. On the axis of symmetry the Kerr metric reduces to

$$ds^2|_{\theta=0} = \left(1 - \frac{2Mr}{r^2+a^2}\right) dt^2 - \left(1 - \frac{2Mr}{r^2+a^2}\right)^{-1} dr^2 \; . \tag{6}$$

It follows that $|\chi| = \sqrt{g}_{tt}$ which is a function only of the r coordinate, further simplifying $\tilde{\kappa} = \sqrt{-g^{rr}}\partial_r|\chi| = \sqrt{g_{tt}}\partial_r\sqrt{g_{tt}} = \frac{1}{2}\partial_r g_{tt}$. A convenient way to compute g'_{tt} is by using the fact that by definition $\Delta = (r-r_+)(r-r_-) = r^2 + a^2 - 2Mr$ which vanishes on the horizon $r \to r_+$ and rewriting

$$g_{tt} = 1 - \frac{2Mr}{r^2+a^a} = 1 - \frac{r^2+a^2-\Delta}{r^2+a^2} = \frac{\Delta}{r^2+a^2} \ . \tag{7}$$

Thus

$$\begin{aligned}\tilde{\kappa} &= \frac{1}{2}\left(\frac{\Delta}{r^2+a^2}\right)' \\ &= \frac{1}{2}\left(\frac{\Delta'}{r^2+a^2}\right) + O(\Delta) \\ &= \frac{1}{2}\left(\frac{(r-r_+)'(r-r_-)}{r^2+a^2}\right) + O(\Delta) \\ &= \frac{1}{2}\left(\frac{r-r_-}{r^2+a^2}\right) + O(\Delta) \ .\end{aligned} \tag{8}$$

Here we are using the fact that we will eventually evaluate the expression on the horizon. For the second equality above this means that we do not have to keep track of the derivative of the denominator since it multiplies a Δ. Similarly, in the third equality, taking into account the derivative of the term $(r-r_-)$ is unnecessary since its pre-factor is $(r-r_+)$ which again vanishes.

At this point we can safely take the limit of (8) as we approach the horizon. The resulting $(r_+ - r_-) = 2\sqrt{M^2-a^2}$ can be rewritten as $2\sqrt{M^2-a^2} = 2(r_+ - M)$. Thus, we obtain the desired result

$$\kappa = \lim_{r\to r_+}\tilde{\kappa} = \frac{r_+ - M}{r_+^2 + a^2} \ . \tag{9}$$

In the Schwarzschild limit $a \to 0$ and $r_+ \to 2M$ so that $\kappa \to 1/4M$. In the extremal limit by contrast, $a \to M$ and $r_+ \to M$ resulting in $\kappa \to 0$.

2.c The pullback of the Kerr metric to the spheres of constant radius is

$$-ds^2| = \Sigma d\theta^2 + \frac{A\sin^2\theta}{\Sigma}d\phi^2 \ . \tag{10}$$

The determinant of this induced metric is $\gamma = A\sin^2\theta \to [(r_+^2+a^2)\sin\theta]^2$ on the horizon so the volume is

$$\text{Area} = \int\sqrt{\gamma}d\theta d\phi = (r_+^2+a^2)\int\sin\theta d\theta\phi = 4\pi(r_+^2+a^2) \ . \tag{11}$$

2.d Varying the area formula we get $\delta A = 8\pi(r_+\delta r_+ + a\delta a)$. Then

$$\frac{1}{8\pi}\delta A = r_+\left[1 + \frac{M}{\sqrt{M^2-a^2}}\right]\delta M + a\left[1 - \frac{r_+}{\sqrt{M^2-a^2}}\right]\delta a$$

$$
\begin{aligned}
&= r_+ \left[\frac{\sqrt{M^2 - a^2} + M}{r_+ - M} \right] \delta M + a \left[\frac{\sqrt{M^2 - a^2} - r_+}{r_+ - M} \right] \delta a \\
&= \frac{r_+^2 \delta M - aM\delta a}{r_+ - M} \ , \qquad (12)
\end{aligned}
$$

where we have used the fact that $\sqrt{M^2 - a^2} = r_+ - M$ repeatedly. From (9) and $r_+^2 + a^2 = 2Mr_+$ we get

$$
\begin{aligned}
\frac{\kappa}{8\pi} \delta A &= \frac{r_+^2}{r_+^2 + a^2} \delta M - \frac{aM}{r_+^2 + a^2} \delta a \\
&= \frac{r_+}{2M} \delta M - \frac{a}{2r_+} \delta a \ . \qquad (13)
\end{aligned}
$$

Substituting $J = aM \Rightarrow \delta a = M^{-1}\delta J - JM^{-2}\delta M$ and using (5) we find

$$
\frac{\kappa}{8\pi} \delta A = \frac{1}{2M} \left(r_+ + \frac{aJ}{Mr_+} \right) \delta M - \Omega_H \delta J \ , \qquad (14)
$$

and it remains to show that the δM pre-factor is equal to 1. Indeed, $J = aM$ and the term in parentheses becomes $(r_+ + a^2/r_+) = (r_+^2 + a^2)/r_+ = (2Mr_+)/r_+ = 2M$.

¿From $E \sim GM^2/L$ we have that G has the units of $ELM^{-2} = LM^{-1}$. The angular momentum has units $[J] = LM$. As $[\Omega_H] = L^{-1}$ and $[\kappa] = L^{-1}$ from problems **2.a** and **2.b**, we find that the $\Omega\delta J$ term has dimensions of mass as does the δM term. The $\kappa\delta A$ term on the other hand has the units of length. Therefore, we must multiply this term by the factor $1/G$ to get

$$
\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega_H \delta J \ . \qquad (15)
$$