Homework 2, Physics 776, Spring 2005
Due Thursday Feb. 24, at the beginning of class.

1. If the metric components are independent of a particular coordinate $x^{\hat{\alpha}}$ in a particular coordinate system, then the corresponding vector field $\xi^a = (\partial_{\hat{\alpha}})^a$ is called a *Killing vector field*. Such a coordinate system is said to be "adapted" to ξ. Show that the coordinate invariant condition for a vector field ξ^a to be a Killing field is

$$\nabla_{(a}\xi_{b)} = 0, \tag{1}$$

where the parentheses represent index symmetrization, and $\xi_b = g_{bc}\xi^c$ as usual.(*Hint*: Use the fact that $\nabla_a g_{bc} = 0$, and evaluate (1) in a coordinate system adapted to ξ^a.)

2. The Kerr metric in Boyer-Lindquist coordinates is

$$ds^2 = (1 - \frac{2Mr}{\Sigma})\, dt^2 + \frac{4Mra\sin^2\theta}{\Sigma}\, dtd\phi - \frac{\Sigma}{\Delta}\, dr^2 - \Sigma\, d\theta^2 - \frac{A\sin^2\theta}{\Sigma}\, d\phi^2 \tag{2}$$

with

$$\begin{aligned} \Sigma &= r^2 + a^2\cos^2\theta & (3)\\ \Delta &= r^2 - 2Mr + a^2 & (4)\\ A &= (r^2 + a^2)^2 - a^2\Delta\sin^2\theta. & (5) \end{aligned}$$

   (a) The angular velocity Ω_H of the horizon is defined by the condition that the Killing vector $\chi = \partial_t + \Omega_H\partial_\phi$ coincides with the null normal to the horizon. Show that

$$\Omega_H = \frac{a}{r_+^2 + a^2} = \frac{a}{2Mr_+} \tag{6}$$

   (where $r_+ = M + (M^2 - a^2))$. *Suggested method*: impose the condition that χ is orthogonal to ∂_ϕ. (Being the null normal to the horizon, it must also be orthogonal to ∂_t, and to itself.) Note that Ω_H is constant over the horizon, as it must be if χ is to be a Killing vector.

   [*continued...*]

(b) One way to define the *surface gravity* κ of a black hole is the magnitude of the gradient of the norm of the horizon-generating Killing vector, evaluated at the horizon. That is,

$$\kappa = |(\nabla_a |\chi|)|_{\mathcal{H}} = (-g^{ab}\, \nabla_a |\chi|\, \nabla_b |\chi|)^{1/2}|_{\mathcal{H}}. \tag{7}$$

Although the Boyer-Lindquist time coordinate goes bad at the horizon, κ can be evaluated this way if one is careful to take a limit approaching the horizon. Even this fairly simply formula is complicated to evaluate, but it simplifies tremendously at the axis of symmetry, since the $\sin^2\theta$ terms can be dropped. Evaluate the surface gravity from this formula on the axis, showing that

$$\kappa = \frac{r_+ - M}{r_+^2 + a^2}. \tag{8}$$

(To see that κ is indeed constant on the horizon, one could also evaluate it at any other θ, or one could use a general argument showing that it is constant.) What does κ become in the Schwarzschild limit? In the extremal limit?

(c) Show that the surface area of the Kerr horizon is $4\pi(r_+^2 + a^2)$.

(d) The area A of a Kerr horizon is a function of the black hole mass and angular momentum. Derive the first law of rotating black hole mechanics governing variations from one Kerr solution to another just by varying these extensive parameters,

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ \tag{9}$$

where $J = aM$ is the angular momentum of the hole. Do the dimensional analysis to determine the factors of G needed in the κdA term if it is to have the dimensions of energy (keeping as usual $c = 1$).