Homework 1 solution, Physics 776, Spring 2005

Painlevé-Gullstrand coordinates

The line element for the unique spherically symmetric, vacuum solution to the Einstein equation can be written as

$$ds^2 = dT^2 - (dr + \sqrt{\frac{2M}{r}}dT)^2 - r^2 d\Omega^2 \qquad (1)$$

Note that a surface of constant T is a flat Euclidean space.

1. Which value of r corresponds to the event horizon? Give a clear and precise explanation of your answer, using the properties of the metric extracted directly from the above expression (i.e. without reference to some other coordinate system, for example).

To eliminate clutter let's adopt units with $2M = 1$. It is convenient to first expand out the square:

$$ds^2 = (1 - r^{-1})dT^2 - 2r^{-1/2} dT\,dr - dr^2 - r^2 d\Omega^2.$$

Let me give two different ways to locate/define the horizon in this context. (a) The surface $r = 1$ is null: the angular directions on it are spacelike but the T-translation direction is null. Therefore this surface describes an outgoing spherical congruence of light rays whose cross-sectional area remains fixed. (This is called a marginally outer-trapped surface.) (b) No causal signal inside $r = 1$ can exit this region. To see why, recall that a causal curve has $ds^2 \geq 0$. When $r < 1$ the dT^2 term is negative, so a causal curve must have $dT\,dr < 0$. If the curve is future oriented[1], $dT > 0$, so $dr < 0$, i.e. it must go to smaller values of r, and in particular cannot escape to the region with $r > 1$. Thus $r = 1 = 2M$ is a causal horizon.

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[1]This deserves a bit more discussion. Recall that the surfaces $T = constant$ are spacelike, so as T increases they foliate spacetime in one causal direction, which we'll call the future.

2. Find the coordinate transformation relating these coordinates to the usual Schwarzschild coordinates (t, r, θ, ϕ).

We seek a new coordinate $t = t(T, r, \theta, \phi)$ that yields the Schwarzschild line element

$$ds^2 = (1 - r^{-1})dt^2 - (1 - r^{-1})^{-1}dr^2 - r^2 d\Omega^2.$$

Spherical symmetry tells us that t should depend only upon T and r. Since T-translation at fixed (r, θ, ϕ) and t-translation at fixed (r, θ, ϕ) are both symmetries of the geometry, dt must be proportional to dT at fixed r, with an r-independent constant of proportionality. To match the line elements at $r \to \infty$ this constant must be unity, hence we seek t of the form $t = T + h(r)$. To find $h(r)$ substitute $dT = dt - h'dr$ into the Painlevé-Gullstrand (PG) line element and impose the requirement that there is no $dtdr$ term. This implies that $h'(r) = -r^{1/2}/(r - 1)$. When this holds the line element takes the Schwarzschild form. Integration (with Mathematica, in my case) yields $h(r) = -2r^{1/2} + \ln[(r^{1/2} + 1)/(r^{1/2} - 1)]$. Thus,

$$t = T - 2r^{1/2} + \ln\left(\frac{r^{1/2} + 1}{r^{1/2} - 1}\right)$$

The factors of $2M$ can be restored in general units by dimensional analysis.

3. The radial curves with $dr = -\sqrt{\frac{2M}{r}}dT$ are timelike, and T is the proper time along these curves. Show that (a) these curves are geodesics which are asymptotically at rest at infinity, and (b) they are orthogonal (in the sense of the spacetime metric) to the surfaces of constant T.

As $r \to \infty$ these curves have $dr/dT \to 0$, hence they are asymptotically at rest wrt the black hole as $T \to -\infty$. But are they geodesics? Remember the variational formulation of the geodesic equation for affinely parametrized geodesics:

$$\delta \int \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \, d\lambda = 0,$$

where the overdot means $d/d\lambda$. This is equivalent to the Euler-Lagrange equations $(d/d\lambda)(\partial L/\partial \dot{x}^\alpha) - \partial L/\partial x^\alpha = 0$ for the Lagrangian $L = \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$. For the PG line element this Lagrangian is

$$L = \frac{1}{2}\left[\dot{T}^2 - (\dot{r} + r^{-1/2}\dot{T})^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right]$$

Consider what happens when the Euler-Lagrange equation is evaluated for a radial curve with $(\dot{r} + r^{-1/2}\dot{T}) = \dot{\theta} = \dot{\phi} = 0$. Because these quantities appear quadratically in the Lagrangian, those terms make no contribution to the Euler-Lagrange equations. Hence the r, θ, and ϕ equations are satisfied trivially. The T equation is simply $\ddot{T} = 0$, which is the statement that T is an affine parameter for the geodesic. Hence these curves are indeed geodesics. We already know directly from the line element that T is the proper time along them, so it is no news that T is an affine parameter along them.

One way to see that these curves are orthogonal to the surfaces of constant T is to compute directly the inner product between the tangent vector to one of these curves and any vector lying in the $T = const.$ surface. In the PG coordinate system the tangent vector to the curves is $U^\alpha = (\dot{T}, \dot{r}, 0, 0)$ and a vector in the surface is $V^\alpha = (0, V^r, V^\theta, V^\phi)$. Their inner product is $g_{\alpha\beta}U^\alpha V^\beta = g_{Tr}\dot{T}V^r + g_{rr}\dot{r}V^r = -V^r(r^{-1/2}\dot{T} + \dot{r}) = 0$.
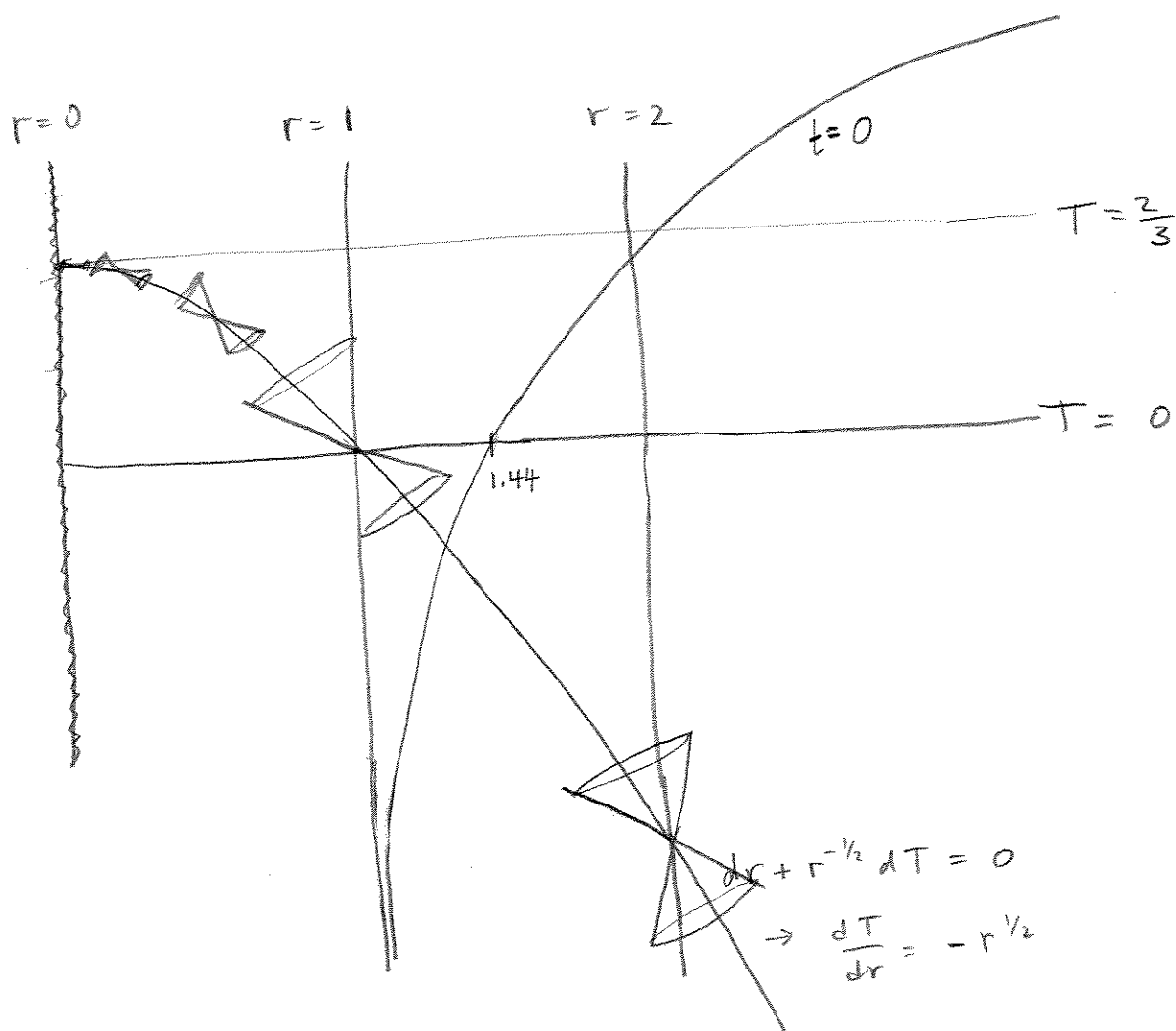
The same conclusion can be reached just by inspection of the line element in the form (1). The angles are irrelevant, so just focus on the two-dimensional (T, r) space at fixed angles. The fact that there is no cross term between dT and $(dr + r^{-1/2}dT)$ in the line element tells us that a displacement on which $dT = 0$ is orthogonal to one on which $(dr + r^{-1/2}dT) = 0$, which means that these geodesics are orthogonal to the surfaces of constant T.

It is worth viewing this more generally. Suppose the metric tensor takes the form

$$g_{ab} = e_a^{(1)}e_b^{(1)} + e_a^{(2)}e_b^{(2)} + e_a^{(3)}e_b^{(3)} + e_a^{(4)}e_b^{(4)} \tag{2}$$

where $e_a^{(i)}$ are four co-vectors. A co-vector defines a three-dimensional "kernel" of vectors with which it has vanishing contraction. Consider the inner product $g_{ab}V^a W^b$ of two vectors V^a and W^b. If V^a is in the kernel of $e_a^{(2,3,4)}$ and W^b is in the kernel of $e_b^{(1)}$ then $g_{ab}V^a W^b = 0$. This is just the situation we have above. The four co-vectors are dT, $(dr+r^{-1/2}dT)$, $rd\theta$, and $r\sin\theta d\phi$. The tangent vector to our geodesics is in the kernel of the last three, while any vector tangent to the surface of constant T is in the kernel of the first.

4. Draw a spacetime diagram of the (r, T) plane showing lines of constant r as vertical and lines of constant T as horizontal. Indicate (a) a radial geodesic $dr = -\sqrt{\frac{2M}{r}}dT$, (b) the light cone at various values of r, and (c) a line of constant Schwarzschild time t.

Diagram labels:

r = 0   r = 1   r = 2   t = 0

$T = \frac{2}{3}$

$T = 0$

1.44

$dr + r^{-\frac{1}{2}} dT = 0$

$\Rightarrow \frac{dT}{dr} = -r^{\frac{1}{2}}$

$dr + r^{-\frac{1}{2}} dT = 0 \Rightarrow r^{\frac{1}{2}} dr = -dT \Rightarrow \boxed{\frac{2}{3} r^{3/2} = T_0 - T}$

arrival time at r = 0.

line of constant Schwarzschild time:

$$T = t + 2r^{\frac{1}{2}} - \ln\left(\frac{r^{\frac{1}{2}} + 1}{r^{\frac{1}{2}} - 1}\right), \text{ with } t = \text{const.}$$

crosses a line of constant PG time T at $2r^{\frac{1}{2}} = \ln\left(\frac{r^{\frac{1}{2}} + 1}{r^{\frac{1}{2}} - 1}\right),$

i.e. $r \cong 1.44$