Hawking Radiation as Tunneling

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May 25, 2005

Abstract

This is a presentation of a paper by Parikh and Wilczek[1] wherein they derive Hawking radiation from particles tunneling through the event horizon of a black hole. [1] The motivation is to provide a more mechanistic or heuristic-friendly derivation of Hawking radiation as a tunneling phenomena.

A key feature of this treatment is that a dynamical geometry is used (the black hole mass is allowed to vary) with conservation of energy strictly enforced. This gives rise to higher order terms in the Hawking radiance! It is thought that a black hole cannot be precisely thermal, because the mass of the black hole changes as it radiates. These higher order terms impose energy corrections for $\frac{\omega}{M}$ not so small, where ω is the energy of the radiated particle and M is the mass of the black hole.

1 Introduction

The common heuristic explanation of Hawking radiation being caused by pair production near the horizon where a negative energy particle falls in and the positive energy particle is radiated out is used as the key mechanism in this derivation of black hole radiation.

Beneath the event horizon there is a space-like killing vector. This allows negative energy states. These states are classically restricted to the interior of the event horizon. But they can tunnel out Quantum Mechanically. This causes pair creation with a positive energy particle outgoing and a negative energy antiparticle ingoing[2].

Parikh and Wilczek consider two possible scenarios. Pair production can occur just inside the horizon with a positive energy particle tunneling out and the pair production can occur just outside the event horizon with a negative energy particle tunneling in.

Classically speaking a particle inside the event horizon of a blackhole is trapped within and cannot escape. We consider a thin shell of energy ω tunneling through the event horizon and escaping to an outgoing geodesic. The

[1] I do not strictly follow the method of Parikh and Wilczek. Instead of taking the WKB result to be true, namely $\psi \propto \exp \frac{i}{\hbar} \int dx P$, I derive the formula beginning with a minimally coupled scalar field. As a result I immediately get the ingoing and outgoing, particle and antiparticle wavefunctions and am able to use fewer tricks initially.

tunneling process will be treated semiclassically with the transmission coefficient determined, via WKB method, from the classical action of the particle.

2 Method

2.1 Nonsingular Coordinates

The first step is to choose coordinates which are not singular across the horizon. A transformation is made from Schwarzschild to Painlevé coordinates. This was our first homework assignment.

Beginning with Schwarzschild coordinates, we shift Schwarzchild time t_S by a function of r, $f(r)$.

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2M}{r}\right)dt_S^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 && (1)\\
t_S &= t + f(r) && (2)\\
ds^2 &= -\left(1-\frac{2M}{r}\right)dt^2 - 2f'(r)\left(1-\frac{2M}{r}\right)dtdr && (3)\\
&\quad + \left(\left(1-\frac{2M}{r}\right)^{-1} - f'(r)^2\left(1-\frac{2M}{r}\right)\right)dr^2 + r^2 d\Omega^2 && (4)
\end{aligned}
$$

Next the metric is made to be spherical for constant time slices. This fixes $f(r)$ and rids us of the singularity.

$$
\begin{aligned}
1 &= \left(1-\frac{2M}{r}\right)^{-1} - f'(r)^2\left(1-\frac{2M}{r}\right) && (5)\\
f'(r)\left(1-\frac{2M}{r}\right) &= \sqrt{\frac{2M}{r}} && (6)\\
ds^2 &= -\left(1-\frac{2M}{r}\right)dt^2 + 2\sqrt{\frac{2M}{r}}dtdr + dr^2 + r^2 d\Omega^2 && (7)
\end{aligned}
$$

2.1.1 Radial, Null Geodesics

With the new metric we can now solve for the only curves that are both radial and null.

$$
\begin{aligned}
0 &= \left(\frac{ds}{dt}\right)^2 && (8)\\
0 &= -\left(1-\frac{2M}{r}\right) + 2\sqrt{\frac{2M}{r}}\dot{r} + \dot{r}^2 && (9)\\
\dot{r} &= \pm 1 - \sqrt{\frac{2M}{r}} && (10)
\end{aligned}
$$

Where the $\pm$ accounts for the outgoing ($\dot{r} > 0$) and ingoing ($\dot{r} < 0$) geodesics when outside the event horizon. When inside the event horizon, both geodesics are ingoing. These curves are obviously null rays in the asymptotically flat space far away from the black hole.

2.2 WKB Approximation

Consider the s-wave of a minimally coupled scalar field for an arbitrary metric. We begin the WKB approximation by casting the field as an exponential and explicitly separating the real and imaginary parts which give the amplitude and the phase.

$$\hbar^2 \nabla^2 \phi = -m^2 \phi \tag{11}$$

$$\phi(x) = e^{T(x)+\imath S(x)} \tag{12}$$

$$\nabla^2 (T + \imath S) + (\nabla T + \imath \nabla S)^2 = -\frac{m^2}{\hbar^2} \tag{13}$$

$$\nabla^2 T + (\nabla T)^2 - (\nabla S)^2 = -\frac{m^2}{\hbar^2} \tag{14}$$

$$\nabla^2 S + 2\nabla S \cdot \nabla T = 0 \tag{15}$$

As this is a semiclassical approximation, we expand both T and S as a power series in $\hbar$. They must start at least with an $\hbar^{-1}$ term or the equations cannot be satisfied.

$$T(x) = \hbar^{-1} \left(T_0(x) + \hbar T_1(x) + \hbar^2 T_2(x) + \cdots \right) \tag{16}$$

$$S(x) = \hbar^{-1} \left(S_0(x) + \hbar S_1(x) + \hbar^2 S_2(x) + \cdots \right) \tag{17}$$

The zero'th order terms are as follows.

$$(\nabla T_0)^2 - (\nabla S_0)^2 = -m^2 \tag{18}$$

$$\nabla T_0 \cdot \nabla S_0 = 0 \tag{19}$$

In the WKB approximation we set the amplitude to be slowly varying as compared to the phase, $\nabla T_0 = 0$.

$$(\nabla S_0)^2 = m^2 \tag{20}$$

Now let us solve for S_0 for a massless field given our choice of metric.

$$g_{\mu\nu} = \begin{bmatrix} -\left(1 - \frac{2M}{r}\right) & \sqrt{\frac{2M}{r}} \\ \sqrt{\frac{2M}{r}} & 1 \end{bmatrix} \tag{21}$$

$$g^{\mu\nu} = \begin{bmatrix} -1 & \sqrt{\frac{2M}{r}} \\ \sqrt{\frac{2M}{r}} & 1-\frac{2M}{r} \end{bmatrix} \tag{22}$$

$$0 = \left(\frac{\partial S_0}{\partial r}\right)^2 - \left(\frac{\partial S_0}{\partial t} - \sqrt{\frac{2M}{r}}\frac{\partial S_0}{\partial r}\right)^2 \tag{23}$$

$$0 = \frac{\partial S_0}{\partial t} + \left(\pm 1 - \sqrt{\frac{2M}{r}}\right)\frac{\partial S_0}{\partial r} \tag{24}$$

$$0 = \frac{\partial S_0}{\partial t} + \dot{r}\frac{\partial S_0}{\partial r} \tag{25}$$

$$S_0 = \pm\omega\left(t - \int^r \frac{dr}{\dot{r}}\right) \tag{26}$$

Where the term in equation [24] we recognize as $\dot{r}$ for the radial, null geodesic.

Via saddle point approximation the semiclassical kernel $\mathcal{K}_{1\to 2}$, that propagates the particle between x_1 and x_2 in configuration space, can now be evaluated.

$$\mathcal{K}_{1\to 2} = \mathcal{N} e^{\frac{i}{\hbar} S_0|_{x_1}^{x_2}} \tag{27}$$

S_0 here acts as the classically evaluated action (note that it does indeed have units of action). Parikh and Wilczek will refer to it as such.

We can use standard results of the WKB method for the calculation of the transmission coefficient for tunneling through a potential barrier that would be forbidden by classical law. The WKB method is used to find solutions before, in, and after the classically forbidden region. The coefficients are matched for continiuty and the resulting transmission coefficient is

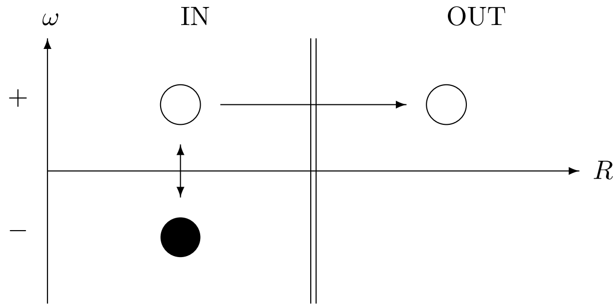
$$\Gamma = e^{-\frac{2}{\hbar}\,\mathrm{Im}\ S|_{x_1}^{x_2}} \tag{28}$$

Where x_1 and x_2 denote the beginning and the end of the classically forbidden region.

2.3 Tunneling

2.3.1 Particle Channel

Consider pair production occurring just beneath the event horizon with the positive energy particle tunneling out. Here is a diagram of the heuristic process.



With respect to the vacuum, this is an occupied negative energy state that is tunneling out causing the pair creation process.[2]

We are looking for the imaginary part of the action over the classically forbidden region.

$$\begin{aligned}
\mathrm{Im}\, S &= \mathrm{Im} \int_{r_{in}}^{r_{out}} dr\ p && (29)\\
\mathrm{Im}\, S &= \mathrm{Im} \int_{r_{in}}^{r_{out}} dr \int_0^p dp' && (30)
\end{aligned}$$

First the classical momentum is expanded into an integral. Next Hamilton's equation is used to transform variables from momentum to energy.

$$\begin{aligned}
\frac{dH}{dp} &= \dot{r} && (31)\\
\mathrm{Im}\, S &= \mathrm{Im} \int_{r_{in}}^{r_{out}} dr \int_M^{M-\omega} dH \frac{1}{\dot{r}} && (32)
\end{aligned}$$

In this last equation we have used the fact that if the particles have tunneled out, then the black hole will have lost some energy ω.

Next we switch integration variables from H to the particle energy ω.

$$\begin{aligned}
H &= M - \omega' && (33)\\
dH &= -d\omega' && (34)\\
\mathrm{Im}\, S &= \mathrm{Im} \int_{r_{in}}^{r_{out}} dr \int_0^{\omega} (-d\omega') \frac{1}{\dot{r}} && (35)
\end{aligned}$$

From my derivation of the WKB solution, we could have come to equation [35] in two steps. Normally it would not occur to expand the energy out into an integral. What this will do is continuously tunnel the particle through the event horizon as opposed to all at once. The authors do not mention this and because of their different route in derivation I am inclined to believe that they may not be aware of it. The affect of this will be correct higher order terms in the Hawking radiance.

$$\mathrm{Im}\ S = \pm\mathrm{Im}\int_{r_{in}}^{r_{out}} dr\frac{\omega}{\dot{r}} \tag{36}$$
$$\mathrm{Im}\ S = \pm\int_0^{\omega} d\omega' \int_{r_{in}}^{r_{out}} dr\frac{1}{\dot{r}} \tag{37}$$

Now because the particle is tunneling out, we use the outgoing null geodesic and what is the closest thing to an outgoing geodesic within the event horizon. It is almost as if there is a classical turning point just beneath the horizon, asymptotically in the infinite past. Though truely these are not classical turning points, but the WKB approximation does not care. The action becomes imaginary and we can match coefficients in the three regions. That is all that matters.

We also take the initial point to be just inside the event horizon. We take the final point to be just outside the event horizon as well, but it must be taken into account that with the black hole losing energy, the event horizon is in a different location after the particle has tunneled out.

Finally, self gravitation of the shell of energy is taken into account replacing M with $M-\omega$ in the metric in which the particle travels. This result is derived by Kraus and Wilczek[3].

$$\dot{r} = +1-\sqrt{\frac{2(M-\omega')}{r}} \tag{38}$$
$$r_{in} = 2M-\epsilon \tag{39}$$
$$r_{out} = 2(M-\omega)+\epsilon \tag{40}$$
$$\mathrm{Im}\ S = -\mathrm{Im}\int_{2M}^{2(M-\omega)} dr\int_0^{\omega} d\omega'\frac{1}{1-\sqrt{\frac{2(M-\omega')}{r}}} \tag{41}$$
$$\mathrm{Im}\ S = \pi\int_0^{\omega} d\omega'\ 4(M-\omega') \tag{42}$$
$$\mathrm{Im}\ S = 4\pi\omega\left(M-\frac{\omega}{2}\right) \tag{43}$$

In equation [41] the contour integral is a right-handed semicircle around the pole by deforming the contour down on the E plane. This gives a prefactor of $\pi\imath$. This gives a positive temperature.

A second calculation will now be performed where the integrals will be evaluated in reverse order. This will reassert the choice of r_{out} as the correct choice.

$$\mathrm{Im}\ S = \mathrm{Im}\int_{r_{in}}^{r_{out}} dr\int_{M}^{M-\omega} dE\frac{1}{1-\sqrt{\frac{2E}{r}}} \tag{44}$$

$$\text{Im } S = \pi \int_{2M-\epsilon}^{r_{out}} dr\,(-r) \tag{45}$$

$$r_{out}^2 = 4M^2 - \frac{2}{\pi} Im(S) \tag{46}$$

$$r_{out}^2 = (2(M-\omega))^2 \tag{47}$$

In equation [44] the contour integral is a right-handed semicircle around the pole by deforming the contour down on the E plane. This gives a prefactor of $\pi\imath$.

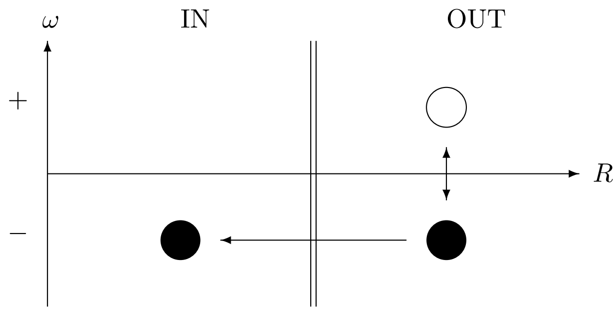
This calculation shows that the value of r_{out} is consistent with the previous calculation of the imaginary part of the action. Note that the first calculation did not depend on precise location of r_{out}, but only that it encompassed a pole along with r_{in}.

Equation [8] of Wilczek and Parikh[1] contains an typesetting error. The last term has both the $\imath$ striped out and Im applied. Doing both sets the term to zero and nullifies the equality.

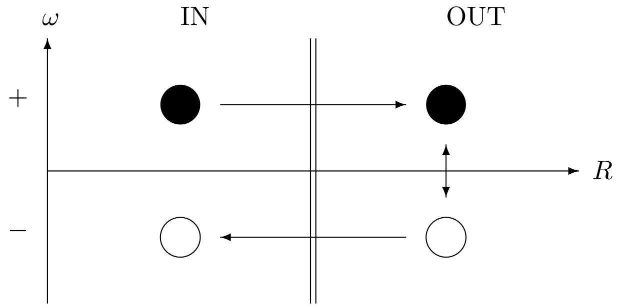
2.3.2 Antiparticle Channel

Now consider pair production occurring just outside the event horizon with the negative energy particle tunneling in, in reverse time. Mathematically this is the exact same process as the first, the integrals will come out to be exactly the same. Physically there is no reason that this would be restricted to antiparticles and the other channel restricted to particles. I think the paper is confusing how negative energy particles are interpreted in QFT with the really negative energy states that can occur when you have a space-like killing vector for t.

This is the reverse-time diagram for the actual calculation that is made.



Here is what I think would actually have to happen, in forward time, for any of this to make sense.



Positive energy must be coming out in forward time for the black hole to be shrinking. The pair creation is also stated to occur outside the event horizon. This leaves the negative energy particle to necessarily fall, not tunnel, but fall into the black hole.

And now on to the calculation. First let us switch to the reversed time metric.

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2M}{r}\right)dt^2 + 2\sqrt{\frac{2M}{r}}dtdr + dr^2 + r^2 d\Omega^2 && (48)\\
t_R &= -t && (49)\\
ds^2 &= -\left(1-\frac{2M}{r}\right)dt_R^2 - 2\sqrt{\frac{2M}{r}}dt_R dr + dr^2 + r^2 d\Omega^2 && (50)\\
\dot{r}_R &= \pm 1 + \sqrt{\frac{2M}{r}} && (51)
\end{aligned}
$$

Now we must choose the incoming geodesic. And we must note to add to the mass of the black hole to take into account the self gravitation of the particle.

$$
\begin{aligned}
\dot{r}_R &= -1 + \sqrt{\frac{2(M+\omega')}{r}} && (52)\\
\mathrm{Im}\, S &= \mathrm{Im} \int_{r_{out}}^{r_{in}} dr \int_{M}^{M+\omega} dH \frac{1}{\dot{r}} && (53)\\
\mathrm{Im}\, S &= \mathrm{Im} \int_{r_{out}}^{r_{in}} dr \int_{0}^{-\omega} (-d\omega') \frac{1}{\dot{r}} && (54)\\
\mathrm{Im}\, S &= -\mathrm{Im} \int_{2M}^{2(M+\omega)} dr \int_{0}^{-\omega} d\omega' \frac{1}{-1+\sqrt{\frac{2(M+\omega')}{r}}} && (55)\\
\mathrm{Im}\, S &= -2\pi \int_{0}^{-\omega} d\omega'\; 2(M+\omega') && (56)\\
\mathrm{Im}\, S &= 4\pi\omega\left(M - \frac{\omega}{2}\right) && (57)
\end{aligned}
$$

2.4 Hawking Radiation

To combine the two contributions Parikh and Wilczek simply add the amplitudes of the Feynman diagrams. However the channels are stated to be for particle and antiparticle, so that wouldn't be allowable. But really both channels could be for either and both. But none of this matters any way because we will get the same exponential term.

$$\begin{aligned} \Gamma &\propto e^{-2 \text{ Im } S_{Total}} && (58) \\ &\propto e^{-8\pi\omega\left(M-\frac{\omega}{2}\right)} && (59) \end{aligned}$$

For small energy ω, this reduces to, $e^{-8\pi M\omega}$, a Boltzmann factor for energy ω at the Hawking temperature $\frac{1}{8\pi M}$

2.4.1 The Higher Order Term

The higher order term enforces energy conservation. Firstly consider the effective temperature $\frac{1}{8\pi\left(M-\frac{\omega}{2}\right)}$. The negative sign ensures that the temperature increases with radiation.

Secondly consider if the emitted particle takes all of the mass of the black hole with it. This would have a transmission rate of

$$\Gamma(\omega = M) \propto e^{-4\pi M^2} \tag{60}$$

There can only be one of these outgoing states. But there are $e^{S_{BH}}$, where S_{BH} is the Berkenstein-Hawking entropy, states in total, so the probability of finding that states is one in that number or

$$\begin{aligned} P(\omega = M) &\propto e^{-S_{BH}} && (61) \\ P(\omega = M) &\propto e^{-\frac{A}{4}} && (62) \\ P(\omega = M) &\propto e^{-4\pi M^2} && (63) \end{aligned}$$

So the higher order correction is in agreement with the Berkenstein-Hawking entropy.

2.5 Charged Black Holes

It is a trivial extension to consider *uncharged* radiation from charged black holes. We begin with the corresponding Panlevé coordinates and then calculate their radial, null geodesics.

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + 2\sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}}dtdr + dr^2 + r^2 d\Omega^2 \tag{64}$$

$$r_H = M \pm \sqrt{M^2 - Q^2} \tag{65}$$

$$\dot{r} = \pm 1 - \sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}} \tag{66}$$

The imaginary part of the action for the outgoing, positive energy particle is

$$\text{Im } S = \text{Im } \int_0^{\omega} (-d\omega') \int_{r_{in}}^{r_{out}} dr \frac{1}{1 - \sqrt{\frac{2(M-\omega')}{r} - \frac{Q^2}{r^2}}} \tag{67}$$

A change of variables before the residue is taken can simplify this integral greatly.

$$u = \sqrt{2(M-\omega')r - Q^2} \tag{68}$$

$$du = \frac{-r}{u} d\omega' \tag{69}$$

$$\text{Im } S = \text{Im } \int_{\sqrt{2Mr-Q^2}}^{\sqrt{2(M-\omega)r-Q^2}} du \int_{r_{in}}^{r_{out}} dr \frac{u}{r-u} \tag{70}$$

$$\text{Im } S = -\pi \int_{M+\sqrt{M^2-Q^2}}^{(M-\omega)+\sqrt{(M-\omega)^2-Q^2}} dr\ r \tag{71}$$

$$\text{Im } S = \pi \left. \frac{r^2}{2} \right|_{(M-\omega)+\sqrt{(M-\omega)^2-Q^2}}^{M+\sqrt{M^2-Q^2}} \tag{72}$$

$$\text{Im } S = \pi \left(2\omega \left(M - \frac{\omega}{2} \right) \right. \tag{73}$$

$$\left. - \left((M-\omega)\sqrt{(M-\omega)^2 - Q^2} - M\sqrt{M^2 - Q^2} \right) \right) \tag{74}$$

$$\Gamma \propto e^{-2 \text{ Im } S} \tag{75}$$

$$\Gamma \propto e^{-2\pi \left(2\omega \left(M - \frac{\omega}{2} \right) - \left((M-\omega)\sqrt{(M-\omega)^2-Q^2} - M\sqrt{M^2-Q^2} \right) \right)} \tag{76}$$

This equation [76] is incorrectly multiplied by 2 in Parikh and Wilczek.

In the next step we will taylor expand the exponent and take the linear terms to find the Hawking temperature. We have

$$\Gamma = e^{-\beta\omega + \alpha_2\omega^2 + \alpha_3\omega^3 + \cdots} \tag{77}$$

$$\beta_H = 2\pi \frac{\left(M + \sqrt{M^2 - Q^2} \right)^2}{\sqrt{M^2 - Q^2}} \tag{78}$$

2.5.1 Higher Order Terms

A similar analysis of the higher order terms was not given in the paper. I attempting to calculate the shell energy ω associated with a probability of 1 in the number of states, one will find that it must be the case that $Q = 0$ or the conditions cannot be satisfied. Such a process of uncharged radiation from a charged black hole cannot happen.

3 Discussion

3.1 Results

Using the humble tools of WKB tunneling, the Hawking temperature is derived for charged and uncharged black holes. The dynamical geometry, along with continuous tunneling reveals higher order terms which enforce energy conservation and are in agreement with the Berkenstein-Hawking entropy.

3.2 Further Research

It may or may not be important to note that the limit of ω comparable to M has been investigated, and yet the particle is treated semiclassically while the metric is only treated classically. Perhaps it would be wise to consider using the Wheeler-DeWitt equation, combining the Hamiltonians for the metric and particle, given that the metric can only vary by M.

I have just begone the most rudimentary investigation and have discovered this so far. The Painlevé coordinates already naturally foliate space and time in the manner needed to use the WDW equation. However the 3-metric q_{ij} is simply the flat spherical metric. The Hamiltonian constraint is identically zero even though you may vary M in time. What is lost are the surface integrals from the action. This contributes to the WDW equation in a most curious manner. This has all been investigated before by Teitelboim and I am just beginning to read his papers.

References

[1] Maulik K. Parikh and Frank Wilczek
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[2] Thibaut Damour and Remo Ruffini
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[3] Per Kraus and Frank Wilczek
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