Overcharging and Overspinning a Black Hole

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Abstract: This is a report on past and current ideas about overcharging and/or overspinning a black hole. The idea of overcharging a black hole is useful in examining the validity of cosmic censorship and exploring the possibilities of at least a mathematical viable method of having a black hole without an event horizon. Since a black hole can only be qualified by its mass M, charge Q, and angular momentum J, the question hinges on an analysis of thermodynamic changes when matter of mass m, charge q, and extrinsic angular momentum j, is added to the black hole. A review of two analyses of this idea will be given as well as some speculation about extending the analysis.

Mathematical Background

The idea of cosmic censorship was first put forward by Penrose [1]. Simply stated, it says that there exist no black holes with out an event horizon, there are no "naked singularities". The veracity of the statement has been under discussion for quite some time and may only really be proven by future space exploration. However, that has not stopped further analysis of the problem to determine if there are any theoretical limits or restrictions to support the idea. One place to start is by analyzing a metric and seeing if a condition exists in which the event horizon, as a natural property of the metric, is not allowed as a physical solution.

The most general solution of a charged, spinning black hole is given by the Kerr-Newman metric:

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a^2 \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{\rho^2}[adt - (r^2 + a^2)d\phi]^2 + \rho^2(\frac{dr^2}{\Delta} + d\theta^2)$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta$$

$$\Delta = r^2 - 2Mr + a^2 + Q^2$$

The quantities M, a=J/M, and Q represent the mass, angular momentum per unit mass, and charge respectively. When a=0, the metric reduces to the Reissner-Nordstrom metric for a charged black hole and when Q=0 also, the final metric is simply Schwarzschild. The condition for an event horizon is given as where the g_{rr} term in the metric is singular. This is when the denominator of that term goes to zero and then gives values of the radii from a quadratic equation. The solutions are

$$r_+ = M + \sqrt{M^2 - (a^2 + Q^2)}$$

$$r_- = M - \sqrt{M^2 - (a^2 + Q^2)}$$

These solutions are given in terms of natural units (G = c = ħ = e = 1), so that everything is in terms of mass. The first solution, $r_{+,}$ is usually associated with the event horizon and will also be denoted by R_H.

So if $M^2 < (a^2 + Q^2)$, the solution for R_H does not have a real-valued solution and the horizon does not exist. The question then becomes how do you increase Q and J? The basic idea is just throw matter into the black hole. By use of the laws of black hole thermodynamics, we can show that the increase of Q and J by δQ and δJ comes at the price of increasing M by δM and thus A by a certain amount through R_H. Starting with the 1st law of BH thermodynamics

$$\frac{1}{8\pi}\kappa\delta A = \delta M - \Omega\delta J - \Phi_{bh}\delta Q$$

This equation from Bardeen, Carter, and Hawking [2], relates changes in M, Q, and J to the change in the area, A, of a black hole. The angular velocity of the horizon is Ω and the electrostatic potential of the horizon is Φ_{bh} ($= -A_a\xi^a$, and ξ^a is the horizon Killing field). The area of the black hole is given by $4\pi R_H$ but R_H is a function of M, Q, and J and the calculation of changes can seem a bit circular. One can use the Raychaudhuri equation which can describe how areas change with time. Things can be simplified if the black hole is in a state where any change in certain parameters may destroy the horizon. Such states are called extremal and are represented by Q=M, a=M, $(a^2 + Q^2) = M^2$ in the Reissner-Nordstrom, uncharged Kerr, and charged Kerr-Newman solutions.

Historical Review

Unfortunately, one cannot just simply "drop" particles of δM=m, δQ=q, and δJ=j into a black hole and see what happens (literally and figuratively). There has not been a lot of specific work on the problem of overcharging or overspinning a black hole. A lot of research could be considered as "indirectly" working on it because of the dependence of the analysis not only on the 1st law of black hole thermodynamics but also on more precise calculations of interactions of black holes and charged particles. One reason I believe there has not been any directly work is because of the directness of one of the first analyses of the problem done by Wald. We will go through a brief overview of his work and then a deeper look into Wald's analysis later in the paper. .

In general terms, Wald [3] and later Semiz [4] looked at adding infinitesimal amounts of charge and angular momentum to a black hole and have shown that the event horizon cannot be destroyed in such a manner. Wald looked an extremal Reissner-Nordstrom (Q=M) black hole where sending in a particle of mass m and charge q would made the final Q>M. The most noticeable effect is due to the charge of the black hole. Even though higher multi-pole orders of electric charge must be radiated away, the black hole has an electric monopole profile outside the horizon so that any matter similarly charged would be electrically repulsed by the black hole. Thus for the particle to cross the event horizon, q>m or the electric repulsion overcomes the gravitational attraction. Increasing the velocity doesn't help because increasing the velocity increases the final mass-energy of the black hole and a contrary condition of q<m is found, preventing overcharging. Wald also found that by increasing the angular momentum of a particle trying to enter an extreme Kerr black hole (a =M), the particle will 'miss' the black hole

due to having a high impact parameter. Semiz using similar methods showed that there will be no first-order increases of Q^2 and a^2. This means that the very most, the mass of the black hole will increase just enough to keep it extremal for the new values of Q and a. These calculations are base on energy and angular momentum arguments and presuming monopole charges of the black hole. Wald has continued to come back to the concept of cosmic censorship and destroying the horizon of charged, spinning black holes over the years. He has an article that is updated every few years about this topic [5]. In the most recent version, he talks about cosmic censorship and horizon destruction in terms of the stability of the black hole solutions to linear perturbations. He has a list of analyses done over the years and what some of the new research in the area is about. One new line of thought is black hole solutions in DeSitter space and what they say about cosmic censorship.

Wald Analysis

The basic analysis by Wald goes as follows. The equation of motion for a charged particle of mass m and charge q is

$$\frac{d^2x^\mu}{ds^2} = \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} = \frac{q}{m}F^{\mu\nu}\frac{dx_\nu}{ds}$$

with Lagrangian

$$L = \frac{1}{2}mg_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + qA_\mu\frac{dx_\nu}{ds}$$

And the constants of motion are

$$p_t = \frac{\partial L}{\partial \dot{t}} = mg_{t\mu}\frac{dx^\mu}{ds} + qA_t$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mg_{\phi\mu}\frac{dx^\mu}{ds} + qA_\phi$$

The vector potential which generates the electromagnetic fields of the black hole is

$$\vec{A} = -\frac{Qr}{\rho^2}(dt - a\sin^2\theta d\phi)$$

Evaluating the constants at infinity,

$$p_t = -E_\infty$$

$$p_\phi = L_\infty$$

These numbers just give values to the free energy and free angular momentum of the particle at infinity with respect to the black hole.

Continuing with the analysis, the time-like nature of the particle's four-velocity means

$$-1 = u^\mu u_\mu = \frac{1}{m^2}g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu)$$

And expanding,

$$-m^2 = g^{tt}(-E_\infty - qA_t)^2 + 2g^{t\phi}(-E_\infty - qA_t)(p_\phi - qA_\phi) + g^{\phi\phi}(p_\phi - qA_\phi)^2 + g^{rr}p_r^{\;2} + g^{\theta\theta}p_\theta^{\;2}$$

Solving for E_∞,

$$E_\infty = -qA_t + \frac{1}{g^{tt}}\left\{ g^{t\phi}(p_\phi - qA_\phi) - \sqrt{\begin{matrix}(g^{t\phi})^2(p_\phi - qA_\phi)^2 - g^{tt}g^{\phi\phi}(p_\phi - qA_\phi)^2 \\ -g^{tt}(g^{rr}p_r{}^2 + g^{\theta\theta}p_\theta{}^2 + m^2)\end{matrix}} \right\}$$

$$E_\infty = -qA_t + \frac{1}{g^{tt}}\left\{ g^{t\phi}(p_\phi - qA_\phi) - \sqrt{(g^{t\phi})^2(p_\phi - qA_\phi)^2 - g^{tt}g^{\phi\phi}(p_\phi - qA_\phi)^2 - g^{tt}(g^{rr}p_r{}^2 + g^{\theta\theta}p_\theta{}^2 + m^2)} \right\}$$

Since this is a quadratic solution, and the determinant is given to be positive, the energy is bounded below:

$$\begin{aligned} E_\infty &\geq -qA_t + \frac{1}{g^{tt}}(g^{t\phi})(p_\phi - qA_\phi) \\ &= \frac{Qqr}{\rho^2} + \frac{a(r^2 + a^2 - \Delta)(L_\infty - Qqra\sin^2\theta/\Sigma)}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta} \end{aligned}$$

If we look at an extremal solution ($M^2 = a^2 + Q^2$) and at the horizon radius ($r = R_H = M + \sqrt{M^2 - (Q^2 + a^2)} = M$), we then get

$$\begin{aligned} E_\infty &\geq \frac{QqM}{M^2 + a^2\cos^2\theta}\left(1 - \frac{a^2\sin^2\theta}{M^2 + a^2}\right) + \frac{a}{M^2 + a^2}L_\infty \\ &\geq \frac{QqM + aL_\infty}{M^2 + a^2} \end{aligned}$$

If we set a=0 and Q=M, in order for Q>M in the final state, then q>m and the electromagnetic forces exceed the gravitational forces and the particle does not go in. Increasing the velocity just increases the left hand side and we would have to have $Q > E_\infty$. If we switch to Q=0 and a=M, then when we compare the angular momentum required to have a>M we find that the impact parameter is so great that the particle totally misses the black hole and does not go in.

However, to have the final state violate the condition of having a horizon ($M^2 < a^2 + Q^2$), we must have (taking a derivative)

$$\begin{aligned} M^2 &< a^2 + Q^2 \\ 2M\delta M &< 2J/M^2\delta J - 2J^2/M^3\delta M + 2Q\delta Q \\ (M + J^2/M^3)\delta M &< J/M^2\delta J + Q\delta Q \\ (M^2 + J^2/M^2)\delta M &< (J/M)\delta J + MQ\delta Q \\ \delta M &< \frac{MQ\delta Q + a\delta J}{M^2 + a^2} \\ E_\infty &< \frac{MQq + aL}{M^2 + a^2} \end{aligned}$$

the exact opposite requirement. Thus by contradiction Wald proves that you can not form a naked singularity with an extremal black hole by this method.

Another Method: Hubeny Analysis

The case of a charged particle has been looked at by Hubeny [6], her basic idea is to use a similar approach with slightly different assumptions. The black hole is a near-extremal Reissner-Nordstrom black hole with no angular momentum, different than Wald's extremal Kerr black hole with charge and angular momentum. Her conditions for destroying the horizon are that 1) the particle falls past the event horizon, 2) the final charge of the black hole is greater than the final mass of the black hole, and 3) the effects of the charged particle interacting the with the black hole's fields, or "backreaction," are negligible. The main concern is the backreaction of the charged particle, stated as the fact that condition (3) may stop condition (1) from being realizable. She does find by applying the first two conditions suitable numbers can be found by presuming the last condition is be met and then arguing that the back reaction is negligible. However, she does conclude that this is not a proof that all three conditions are met simultaneously and better calculation does not guarantee that it can. We shall go through her analysis next.

The metric is the same as given above in the beginning of the paper except in it is now in Eddington-Finkelstein coordinates (v, r, θ, φ):

$$ds^2 = -g(r)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

where

$$g(r) \equiv 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{(r-r_+)(r-r_-)}{r^2}$$

and $r_\pm$ are the solutions representing the horizons with a=0.

The equation of motion of a particle analogous to Wald is

$$u^a \nabla_a u^b = \frac{q}{m} F^b{}_c u^c$$

$$F_{ab} = 2\nabla_{[a} A_{b]}$$

$$A_a(r) = (-\frac{Q}{r}, 0, 0, 0)$$

with F_{ab} representing the electromagnetic field strength tensor and A_a being the electrostatic potential. The particle is presumed to be coming in radially, so there is no angular momentum contribution from the particle either and it 4-velocity is just $u^a = \dot{v}(\partial/\partial v) + \dot{r}(\partial/\partial r)$. ($\dot{v} = \partial v/\partial\tau$, where τ is an affine parameter along the particle's worldline).

The energy is a constant of motion, the "projection" of the "v" Killing vector:

$$E_\infty = E = -(\partial_v)^a(mu_a + qA_a) = m(g(r)\frac{dv}{d\tau} - \frac{dr}{d\tau}) - qA_v$$

Using the timelike nature of the four-velocity, we can say

$$-1 = u^a u_a = -g(r)\left(\frac{dv}{d\tau}\right)^2 + 2\left(\frac{dv}{d\tau}\right)\left(\frac{dr}{d\tau}\right) = -\frac{1}{m}(E + qA_v)\left(\frac{dv}{d\tau}\right) + \left(\frac{dv}{d\tau}\right)\left(\frac{dr}{d\tau}\right)$$

Solving for dr/dτ, we get a radial equation of motion:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{m^2}[E + qA_v(r)]^2 - g(r)$$

Points where dr/dτ =0 are turning points in the particle's trajectory and if the previous equation stays positive outside the horizon at r_+, the particle will fall into the black hole.

Hubeny's bounds on the energy come from the equation of motion constraint being satisfied and the mass-charge inequality that makes up the second condition:

$$E > \frac{Qq}{r_+}$$

$$E < Q + q - M$$

Satisfying both requires

$$q > \frac{r_+ - Q}{2}$$

One can clearly see that for an extremal black hole in this case where $Q = M = r_+$, these equations can not be satisfied, which coincides with Wald analysis.

Finally, to keep the radial equation of motion positive, we must have

$$m < Q\sqrt{\frac{-E^2 + 2\frac{M}{Q}Eq - q^2}{M^2 - Q^2}}$$

Thus the key to the analysis is picking the parameters of the incoming particle (q, E, m) for the parameters (Q, M) of the black hole appropriately (satisfying the above constraints) to overcharge the black hole.

The second part of her argument is that the "backreaction" is negligible. The "backreaction" effects include making sure the particle isn't too big to go into the black hole or appreciatively affect the background metric and that the relativistic and electromagnetic energy are not big enough to affect the curvature. The new equation of motion taking in effects up to q^2/m is

$$\begin{aligned} u^b \nabla_b u^u &= \frac{q}{m} F^{ab} u_b \\ &+ \frac{2}{3}\frac{q^2}{m}\left(\frac{q}{m} u^c \nabla_c F^{ab} u_b + \frac{q^2}{m^2} F^{ab} F_{bc} u^c - \frac{q^2}{m^2} u^a F^{bc} u_c F_{bd} u^d \right) \\ &+ \frac{1}{3}\frac{q^2}{m}\left(R^a{}_b u^b + u^a R_{bc} u^b u^c \right) \\ &+ \frac{q^2}{m} u_b \int_{-\infty}^{\tau} \nabla^{[b} (G^-)^{a]c} u_c(\tau') d\tau' \end{aligned}$$

This new equation comes from a paper by Quinn and Wald [7]. These terms in order describe the original background Maxwell tensor, three Abraham-Lorentz radiation damping terms, two local curvature terms, and a "tail" term which is a correction term. By mathematical analysis, she finds the new equation of motion simplified to give

$$u^b \nabla_b u^a = \frac{q}{m} F^{ab} u_b (1 + \frac{2}{3}\frac{q^2}{m}\left(\dot{r}\frac{f'}{f} + g'\dot{v}\right) + \left(\frac{q^2}{m} F^{ab} u_b\right)\left[-qf'(g\dot{v} - \dot{r})\right]$$

$$f(r) \equiv F^{vr} = \frac{Q}{r^2}, f'(r) = -\frac{2Q}{r^3}$$

$$\dot{*} = \frac{d}{d\tau}, *' = \frac{d}{dr}$$

With these equations, she takes a linear expansion about the parameters M=1 and Q=1-2ε^2 with ε being small and does an expansion about ε. What she finds is that the order of the backreaction effects are the same order of the particle's velocity, namely of $O(\varepsilon)$. This means that it is possible for the backreaction to stop the particle from reaching the horizon. Physically, this can be understood by the fact as the particle gets closer to the horizon, it loses energy and may end up with not enough energy to overcome electrostatic repulsion. To see what really happens, would require solving the previous equation which has two problems: 1) it is a very hard equation to solve and 2) energy is not longer a constant so the equation needs to be adjusted. Hubeny goes through a numerical calculation and finds the particle does not fall into the black hole with the parameters chosen. By relaxing some of the conditions, the particle is found to have enough energy to fall into the hole. Her final conclusion is that more definite work needs to be done.

Other Considerations

One may think that maybe there is the possibility of overspinning a black hole by lowering its mass. The case of a positive spinning black hole was looked at indirectly by Christodoulou in 1970 [8]. He was primarily concerned about reducing the mass of a black hole by addition of angular momentum and extracting energy out by a Penrose process. In his paper the next year with Ruffini, he discusses the reduction of mass of a black hole by the addition of particles with rest mass, μ, and charge, ε. The key conclusion he reached was that there was an *irreducible* component of the black hole was that could not be decreased. This irreducible mass determines the size of the event horizon and simply corresponds to the case of a non-charged, non-spinning (Q,J=0) black hole. This can be seen as a statement that the event horizon can not be destroyed by simply removing mass from the system, or trying to "undermass" a black hole into a naked singularity.

An Extension of the Hubeny Analysis to Angular Momentum

One of the suggestions that Hubeny makes in her conclusions is that this same analysis can be done for an uncharged spinning black hole. Looking at the analyses of Wald and Hubeny certainly suggested a definite method of looking at such a question and maybe generalizing it. Next we will take a look at the possibility of extending the analysis further.

The basic method is to start with the metric and the particle's 4-velocity, define an equation for the particle's motion, get the constants of motion, apply the proper normalization to the particle's 4-velocity, and solve for either bounds on the system or for actual equations of motion.

If one starts with the Kerr-Newman metric and sets the charge Q equal to 0, then there will be no electromagnetic interactions to worry about and simplifies the equations of motion to geodesic equations. But there is no simple coordinate system like Eddington-Finkelstein to simplify the metric with so the problem has to be dealt with in "regular" coordinates (t,r,θ,φ). We can simplify matters some by looking along the equatorial plane θ=π/2. The metric is now

$$ds^2 = -\left(\frac{r^2-2mr}{r^2}\right)dt^2 - \frac{2a(2mr)}{r^2}dtd\phi + \frac{r^2}{r^2+a^2-2mr}dr^2 + r^2 d\theta^2 + \left((r^2+a^2) - \frac{(2mr)a^2}{r^2}\right)d\phi^2 .$$

The particle's 4-velocity must be given as $u^a = \dot{t}(\partial/\partial t) + \dot{r}(\partial/\partial r) + \dot{\phi}(\partial/\partial\phi)$. Note the new angular term which represents the contribution from the particle. Enforcing the normalization of the 4-velocity, $-1 = u^a u_a$, gives us another equation to work with

$$-1 = u^a u_a = g_{ab}u^a u^b = -\left(\frac{r^2-2mr}{r^2}\right)\dot{t}^2 - \frac{2a(2mr)}{r^2}\dot{t}\dot{\phi} + \frac{r^2}{r^2+a^2-2mr}\dot{r}^2 + \left((r^2+a^2) - \frac{(2mr)a^2}{r^2}\right)\dot{\phi}^2$$

Without the electromagnetic fields, the particle has no deviation to geodesic motion and thus has to follow the simple geodesic equation, $u^a \nabla_a u^b = 0$.

The constants of motion are the energy and angular momentum, given by their appropriate Killing vectors projected onto the 4-momentum. But due to the cross terms now in the metric, the constants are now crossed up:

$$E = -\left(\frac{\partial}{\partial t}\right)^a mu_a = mg_{tb}u^b = m\left(\frac{r^2-2mr}{r^2}\right)\dot{t} + \frac{2ma(2mr)}{r^2}\dot{\phi}$$

$$L = -\left(\frac{\partial}{\partial\phi}\right)^a mu_a = -m\left((r^2+a^2) - \frac{2mra^2}{r^2}\right)\dot{\phi} + \frac{2ma(2mr)}{r^2}\dot{t}$$

So now we have a set of equations that theoretically can be solved for the radial and angular motions which can be analyzed to see if the particle can cross the horizon.

From the Hubeny analysis, having a coordinate system like Eddington-Finkelstein would be very helpful in simplifying the mathematics involved by simplifying the metric. The original idea of Eddington and Finkelstein was to use freely falling photons as the foundation of the coordinate system [9]. The coordinate v represents ingoing photon geodesics given by the equation v=constant, where $v=t+r^*$. The r^* coordinate represents a "tortoise coordinate" that fixes the singularity at the horizon for Schwarzschild metrics. It should be possible to start from the same basic idea of using infalling photon geodesics and construct a "nicer" metric involving a spinning black hole. Once that is found, the calculations for the equations of motion for the particles may become tractable to numerical simulation.

One could go further and put the charges back into the metric and equations of motion and follow the same method, now looking at a charged, spinning black hole. I think a clearer insight can be gained by solving the non-charged case first and then applying the methods to the charged case. A major contribution would be another set of infalling photon geodesics equations to construct a "nicer" metric. Also, the geodesic equation for the non-charged case is a general differential equation while in the charged case, the extra term involving the electromagnetic field tensor and 4-velocity, can be considered a "particular" solution for the same differential equation.

Conclusion

I have given an updated review of the progress of overcharging and overspinning black holes. The direct research is sparse but a lot can be inferred and improved by analysis of indirect research on the 1st law of black hole thermodynamics and charged particle interaction with charged, spinning black holes. I have also shown the basic analysis done by Wald and Hubeny on this idea and looked at extending their research. I have also shown the possibility of extending their research to more cases. Hopefully, more direct research will be given on this interesting subject.

[1] R. Penrose, "Gravitational Collapse: the Role of General Relativity," Riv. Nouvo Cim. **1**, 252 (1969)

[2] J. M. Bardeen, B. Carter, S.W. Hawking, Commun. Math. Phys. **31**, 161 (1973).

[3] R. M. Wald, "Gedanken experiments to destroy a black hole", Ann. Physics **82** 548-556

[4] I. Semiz, "Dyon black holes do not violate comsic censorship", Class. Quantum Grav. **7** (1990).

[5] R..M. Wald, "Gravitational Collapse and Cosmic Censorship", arXiv:gr-qc/9710068, v1-v3.

[6] V. Hubeny, "Overcharging a Black Hole and Cosmic Censorship", arXiv:gr-qc/9808043 v1

[7] T. C. Quinn & R. M. Wald, "An Axiomatic Approach to Electromagnetic and Gravitational Radiation Reaction of Particles in a Curved Spacetime," Phys. Rev. D**56**, 3381, (1997), gr-qc/9610053.

[8] Demetrios Christodoulou, "Reversible and Irreversible Transformations in Black-Hole Physics," Phy. Rev. Lett. **25**, 1596 (1970).

[9] C. W. Misner, K. Thorne, J. Wheeler, Gravitation. W. H. Freeman and Co. New York: 1973.