* Hawking & Ellis
"The large scale structure of space-time"

CARTER-PENROSE DIAGRAMS

1 - Conformal transf. & global causal structure

* Conformal transformation : $g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}$

* Fundamental property : null geodesics remain unchanged
  $\Rightarrow$ invariance of causal structure under conf. transf.

Proof : $\tilde{\Gamma}^{\rho}_{\mu\nu} = \frac{1}{2}\, \tilde{g}^{\rho\sigma}\left(\partial_\mu \tilde{g}_{\nu\sigma} + \partial_\nu \tilde{g}_{\mu\sigma} - \partial_\sigma \tilde{g}_{\mu\nu}\right) =$

$$= \Gamma^{\rho}_{\mu\nu} + \cancel{2}\,\cancel{\tfrac{1}{2}}\left[\delta^{\rho}_{\nu}(\partial_\mu \ln\Omega) + \delta^{\rho}_{\mu}(\partial_\nu \ln\Omega) - g_{\mu\nu}\, g^{\rho\sigma}(\partial_\sigma \ln\Omega)\right]$$

$$u^a \tilde{\nabla}_a u^b = u^a \nabla_a u^b + 2u^b (u^a \nabla_a \ln\Omega) - \underbrace{0}_{\text{null}} \overset{\substack{\text{originally}\\ \text{geodesic}}}{=} 2u^b \underbrace{(u^a \nabla_a \ln\Omega)}_{\substack{\text{no longer}\\ \text{affinely parametrized}}}$$

* One can use suitable conf. transf. to bring infinity within a finite distance.
  This is very useful to study global properties of the causal structure for a given spacetime.

• Useful tool : Carter-Penrose diagrams

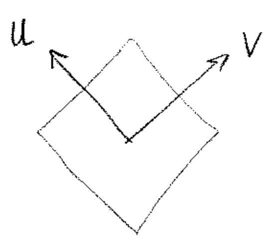
2-D diagrams (projections ; spherical symmetry) $\leftrightarrow$ 1+1 geome[try] is always conformally flat

(radial) light rays at 45°.

• Simplest example : 1+1 Minkowski

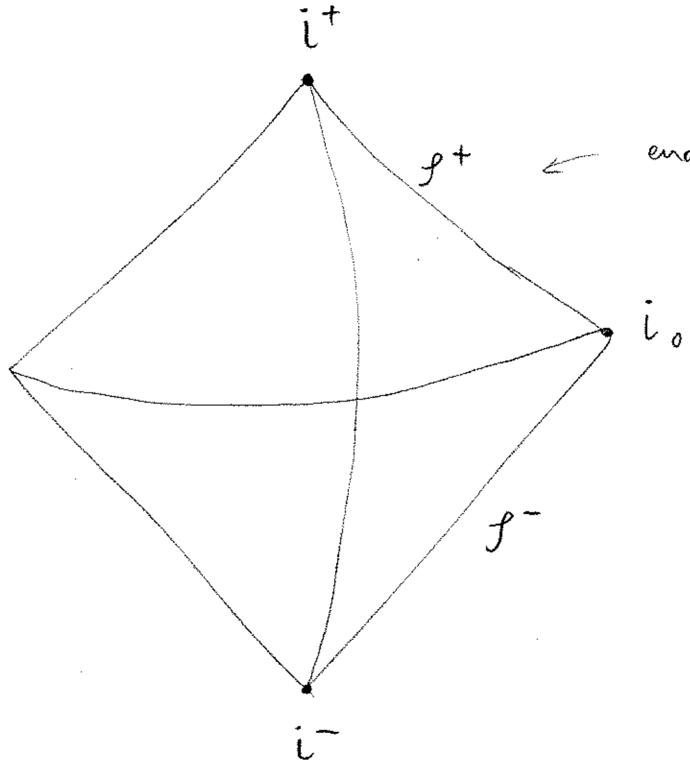
$$ds^2 = -dt^2 + dx^2 = -du\,dv = \sec^2 U \sec^2 V \,(-dU\,dV)$$

$\begin{pmatrix} u = t - r \\ v = t + r \end{pmatrix}$ $\begin{pmatrix} u = \tan U \\ v = \tan V \end{pmatrix}$ $\boxed{\begin{array}{c} -\frac{\pi}{2} < U < \frac{\pi}{2} \\ -\frac{\pi}{2} < V < \frac{\pi}{2} \end{array}}$

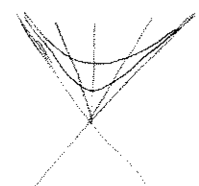


$$= \sec^2(\tilde{t} - \tilde{r}) \sec^2(\tilde{t} + \tilde{r}) \cdot (-d\tilde{t}^2 + d\tilde{x}^2)$$

$\begin{pmatrix} U = \tilde{t} - \tilde{r} \\ V = \tilde{t} + \tilde{r} \end{pmatrix}$

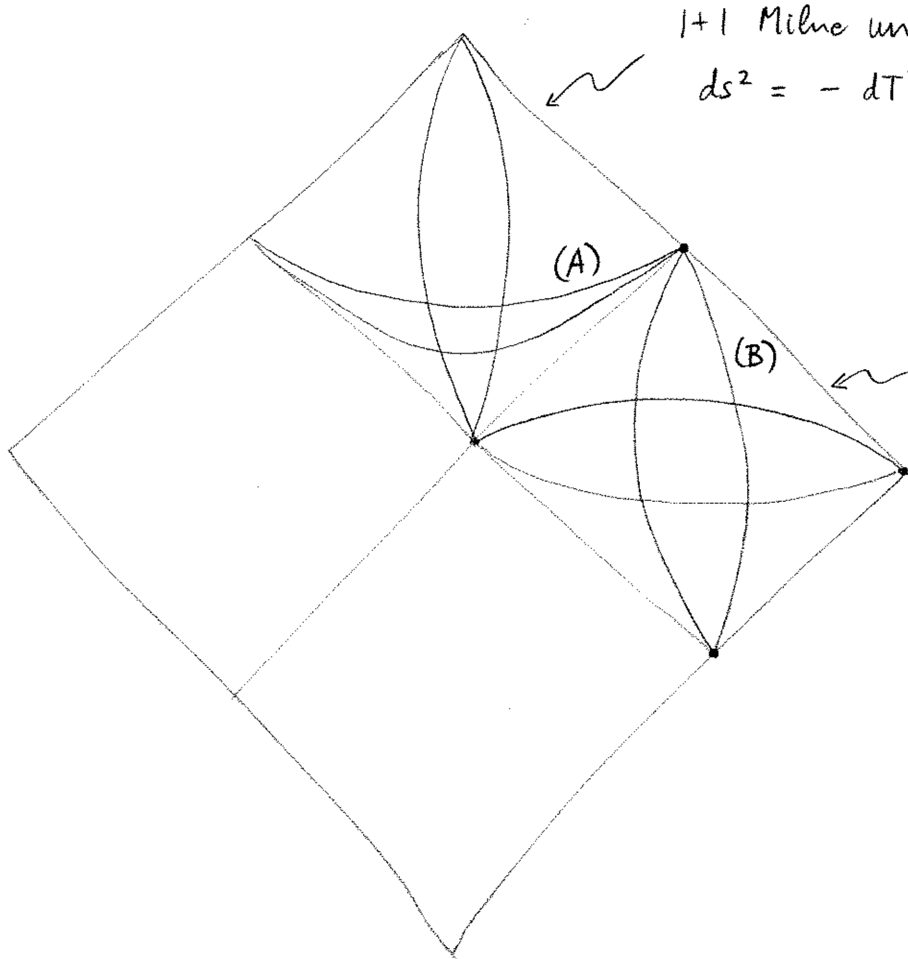


endpoints for
- null (geodesics) curves
- timelike traject. with asympt. velocity c (A)
- asymptotically null spatial curves (B)



1+1 Milne universe

$$ds^2 = -dT^2 + T^2 d\chi^2$$



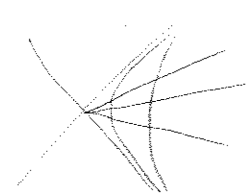
1+1 Rindler wedge

$$ds^2 = -\xi^2 dT^2 + d\xi^2$$

ξ = cnst. → unif. accel. observer

rigid motion

(different accel. for different values of ξ)



2- Examples in 4-D : Minkowski & Spherically symmetric BH's

* 3+1 Minkowski

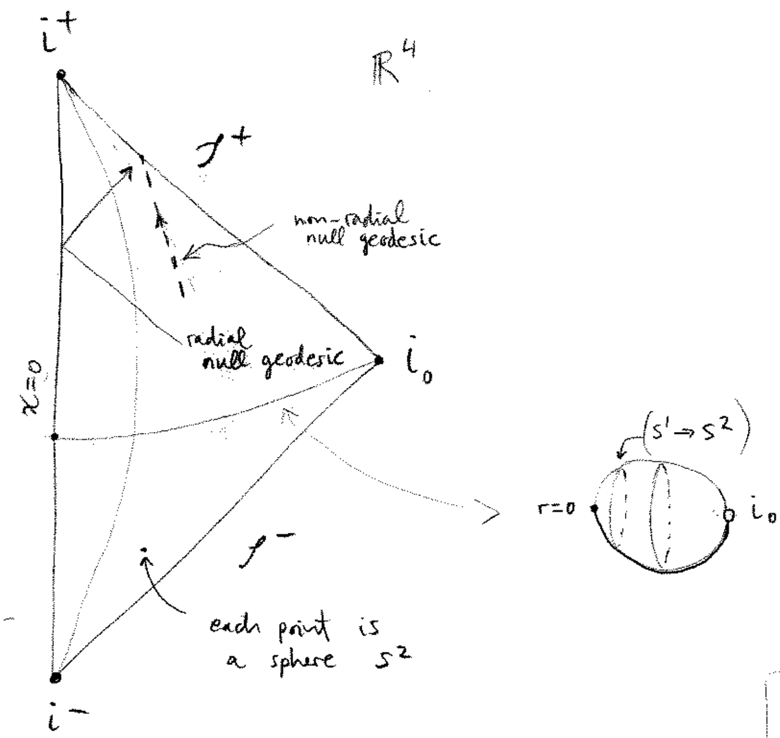
$$ds^2 = -dt^2 + dr^2 + r^2 \underbrace{(d\theta^2 + \sin^2\theta\, d\varphi^2)}_{\equiv\, d\Omega^2} = -du\,dv + \left(\frac{v-u}{2}\right)^2 d\Omega^2 =$$

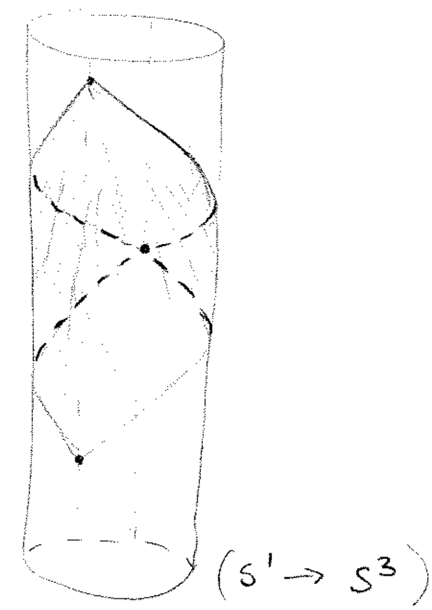
$$= \frac{1}{4}\sec^2 U \sec^2 V \left[-4\,dU\,dV + \sin^2(U-V)\, d\Omega^2\right] =$$

$$= \sec^2(\tilde{t}-\chi)\sec^2(\tilde{t}+\chi)\left[-d\tilde{t}^2 + d\chi^2 + \sin^2\chi\, d\Omega^2\right]$$

$$\left(\begin{aligned}\chi &= U - V\\ \tilde{t} &= U + V\end{aligned}\right)$$

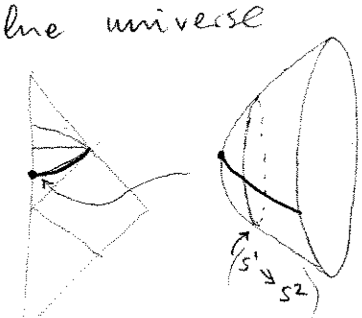
Conformal to the Einstein static universe





- This structure is roughly used to define asymptotic flatness (asymptotically flat regions)

3+1 Milne universe



* 3+1 BH's (Spherically symmetric)

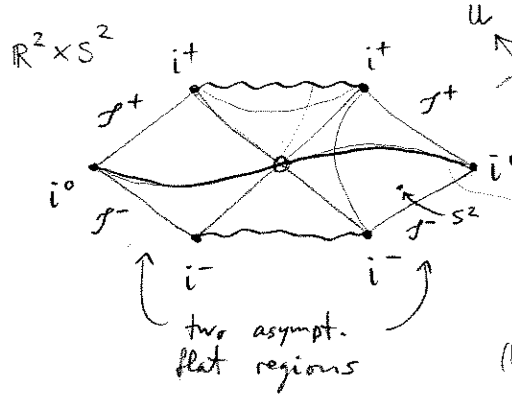
– Schwarzschild : $ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2 d\Omega^2$

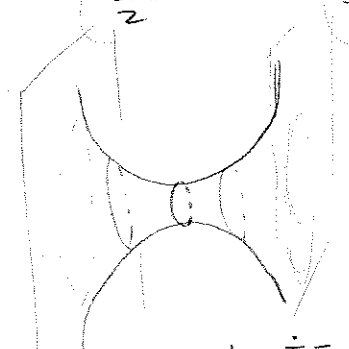
Kruskal maximal extension $\longrightarrow$ $ds^2 = F^2(t',r')\left(-dt'^2+dr'^2\right) + r^2(t',r')\, d\Omega^2$

$$F^2 = \frac{16M^2}{r} e^{-r/2M} \quad ; \quad t'^2 - r'^2 = -(r-2M)\, e^{r/2M}$$

$$\begin{pmatrix} v' = t'+r' \\ u' = t'-r' \end{pmatrix} \longrightarrow \begin{pmatrix} \tan V = \frac{v'}{\sqrt{2M}} \\ \tan U = \frac{u'}{\sqrt{2M}} \end{pmatrix} \qquad -\frac{\pi}{2} < V < \frac{\pi}{2} \;,\; -\frac{\pi}{2} < U < \frac{\pi}{2}$$

$$-\frac{\pi}{2} < U + V < \frac{\pi}{2}$$





$\bar{M}$

(future) Event horizon $= H^+ = \dot{J}^-(\mathscr{I}^+)$

(past) " " $\quad H^- = \dot{J}^+(\mathscr{I}^-)$

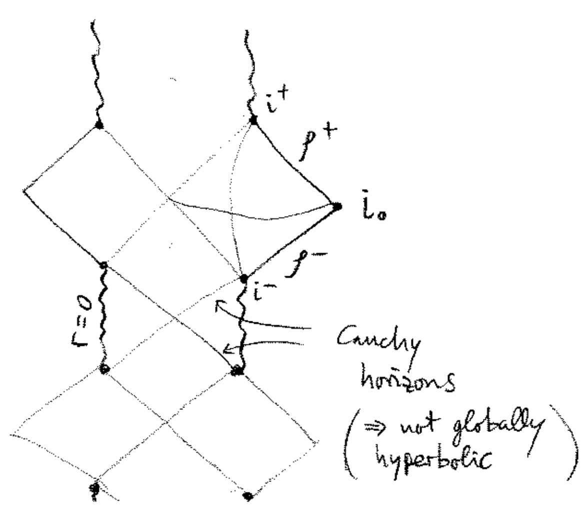
· Cauchy development & global hyperbolicity

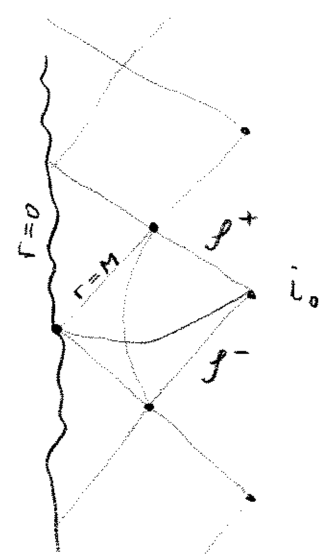
– Reisner – Nordstrom : $ds^2 = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 + \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dr^2 + r^2 d\Omega^2$

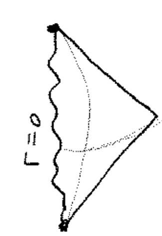
$Q^2 < M^2$

$Q^2 = M^2$ extremal BH

$Q^2 > M^2$ elem. particle







* Cosmological examples (spatially homogeneous & isotropic)

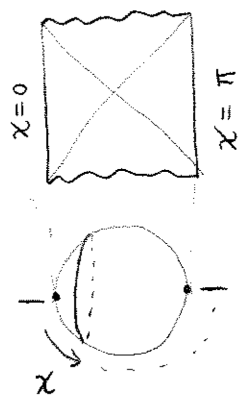
– General <u>R-W metrics</u> :

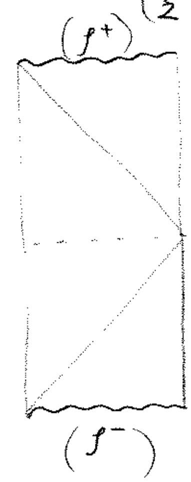
$\boxed{k=1}$ $\quad ds^2 = a^2(\eta)\left[-d\eta^2 + d\chi^2 + \sin^2\chi \, d\Omega^2\right]$

exs. (i) radiation dominated $\qquad$ (ii) matter dominated

$a \propto \sin\eta \qquad\qquad a \propto \sin^2\left(\frac{\eta}{2}\right)$

$R \times S^3$





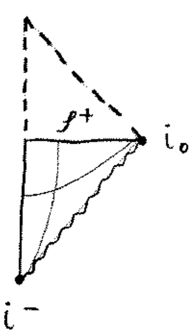
$\boxed{k=0}$ $\quad ds^2 = a^2(\eta)\left[-d\eta^2 + dr^2 + r^2 d\Omega^2\right]$

exs. $\quad p = \gamma\rho$

$\mathbb{R}^4$

$a \propto (-\eta)^{-\left|\frac{2}{1+3\gamma}\right|}$

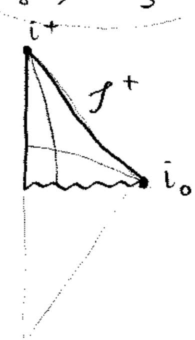
$\gamma < -1/3$



$\gamma = -1/3$
$a \propto e^{\eta}$

$a \propto \eta^{\frac{2}{1+3\gamma}}$

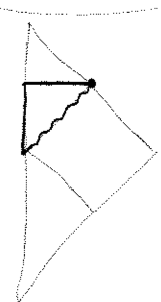
$\gamma > -1/3$



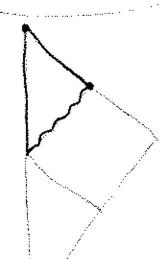
$\boxed{k=-1}$ $\quad ds^2 = a^2(\eta)\left[-d\eta^2 + d\chi^2 + \mathrm{sh}^2\chi \, d\Omega^2\right]$

$\mathbb{R}^4$

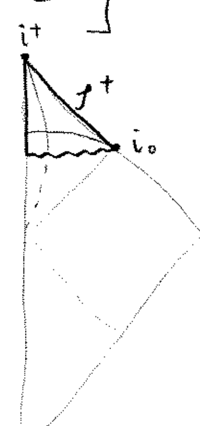
$\gamma < -1/3$



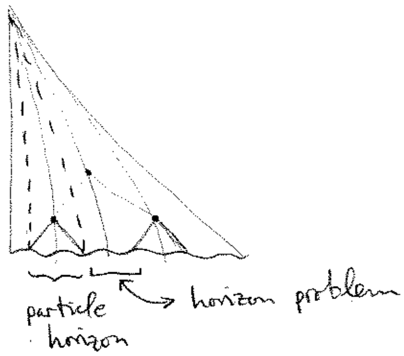
$\gamma = 1/3$

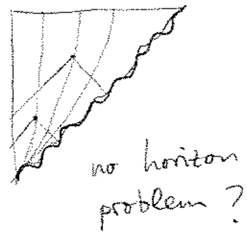


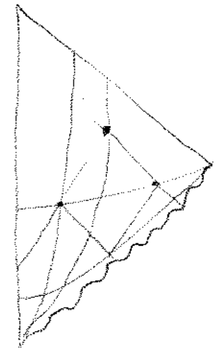
$\gamma > -1/3$



Particle horizon (horizon problem) :







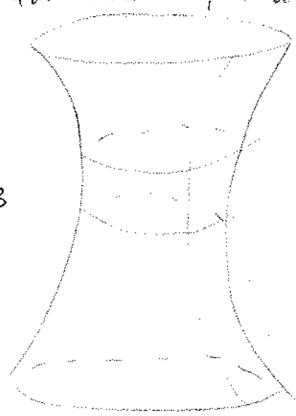
• <u>de Sitter</u> ($\Lambda > 0$, maximally symmetric)

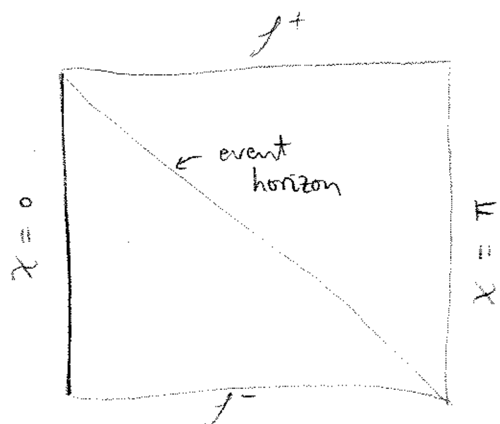
$$-u^2 + v^2 + x^2 + y^2 + z^2 = H^{-2} = \frac{\Lambda}{3} \quad \leftarrow \text{embedding in } 4+1 \text{ Mink.}$$

$$ds_s^2 = -du^2 + dv^2 + dx^2 + dy^2 + dz^2$$

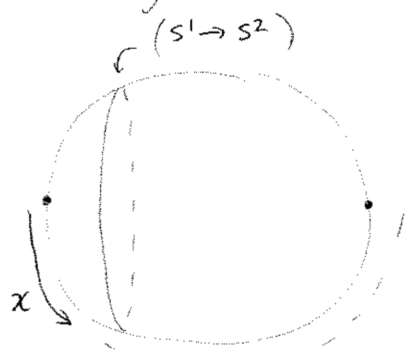
(Geometric units $8\pi G = c = 1$)

$\mathbb{R} \times S^3$





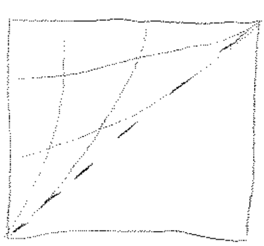
$$ds^2 = \left(\frac{1}{H \sin\eta}\right)^2 \left[-d\eta^2 + d\chi^2 + \sin^2\chi \, d\Omega^2\right]$$



Other coord. systems :

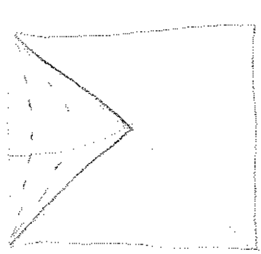
$$ds^2 = \left(\frac{1}{H\eta}\right)^2 \left[-d\eta^2 + dr^2 + r^2 d\Omega^2\right]$$

(flat)



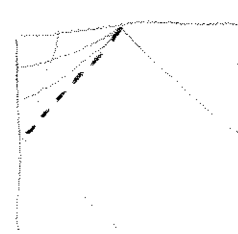
$$ds^2 = -(1 - r^2H^2)\, dt^2 + \frac{dr^2}{(1 - r^2H^2)} + r^2 d\Omega^2$$

(static)



$$ds^2 = -dt^2 + \frac{1}{H^2} \mathrm{sh}^2 Ht \left(d\chi^2 + \mathrm{sh}^2\chi \, d\Omega^2\right)$$

(open)



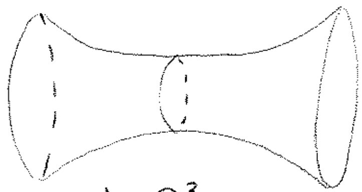
- Anti-de Sitter ($\Lambda < 0$, maximally symmetric)

$$-u^2 - v^2 + x^2 + y^2 + z^2 = -H^{-2} = \frac{\Lambda}{3} \leftarrow \text{embedding in } 3+2 \text{ flat space}$$

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2$$

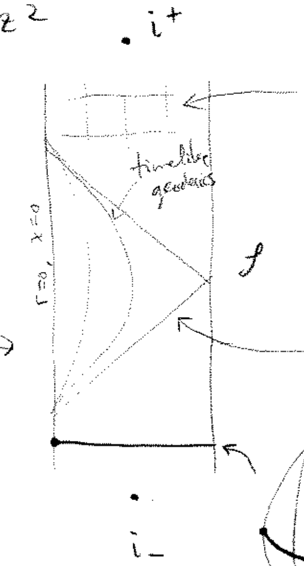
$$\underbrace{ds^2 = -\left(1+\frac{|\Lambda|}{3}r^2\right)dt^2 + \left(1+\frac{|\Lambda|}{3}r^2\right)^{-1}dr^2 + r^2 d\Omega^2}$$

$$ds^2 = -ch^2\tilde{r}\, dt'^2 + d\tilde{r}^2 + sh^2\tilde{r}\, d\Omega^2$$



$S^1 \times \mathbb{R}^3$

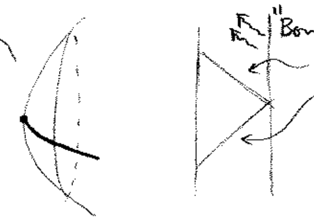
universal covering space (simply connected) $\mathbb{R}^4$



(Alternatively,

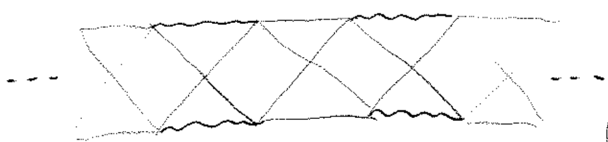
)

$$ds^2 = -dt^2 + \cos^2 t\left(d\chi^2 + sh^2\chi\, d\Omega^2\right)$$



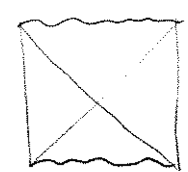
* Further examples :

- Schwarzschild-dS

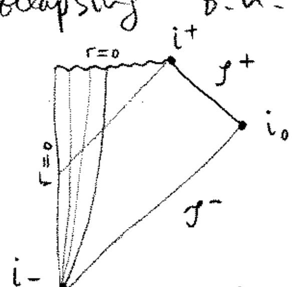


$$ds^2 = -\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)dt^2 + \left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2 d\Omega^2$$

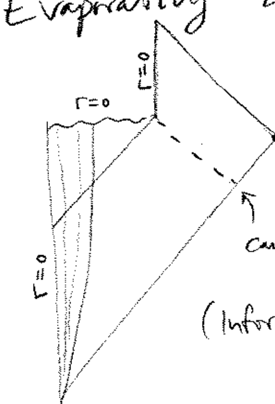
- Schwarzschild-adS



- Collapsing b.h.



- Evaporating b.h.



(Information paradox)

- Inflation ($\Lambda > 0$) + flat FRW ($\gamma > -1/3$)