

* Hawking & Ellis
The large scale structure
of space-time

CARTER - PENROSE DIAGRAMS

1 - Conformal transf. & global causal structure

* Conformal transformation: $g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}$

* Fundamental property: null geodesics remain unchanged
 \Rightarrow invariance of causal structure under cont. transf.

Proof: $\tilde{\Gamma}_{\mu\nu}^\rho = \frac{1}{2} \tilde{g}^{\rho\sigma} (\partial_\mu \tilde{g}_{\nu\sigma} + \partial_\nu \tilde{g}_{\mu\sigma} - \partial_\sigma \tilde{g}_{\mu\nu}) =$
 $= \Gamma_{\mu\nu}^\rho + \frac{1}{2} \left[\delta_{\nu}^\rho (\partial_\mu \ln \Omega) + \delta_\mu^\rho (\partial_\nu \ln \Omega) - g_{\mu\nu} g^{\rho\sigma} (\partial_\sigma \ln \Omega) \right]$

$u^a \tilde{\nabla}_a u^b = u^a \nabla_a u^b + 2u^b (u^a \nabla_a \ln \Omega) - \overset{\substack{\text{originally} \\ \text{geodesic}}}{0} = 2u^b (u^a \nabla_a \ln \Omega)$
 \uparrow
null \uparrow
no longer affinely parametrize.

* One can use suitable cont. transf. to bring infinity within a finite distance. This is very useful to study global properties of the causal structure for a given spacetime.

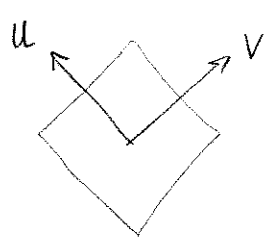
• Useful tool: Carter - Penrose diagrams

2-D diagrams (projections; spherical symmetry) \leftrightarrow (radial) light rays at 45° .

1+1 geom. is always conformally flat

• Simplest example: 1+1 Minkowski

$ds^2 = -dt^2 + dx^2 = -du dv = \sec^2 u \sec^2 v (-du dv)$

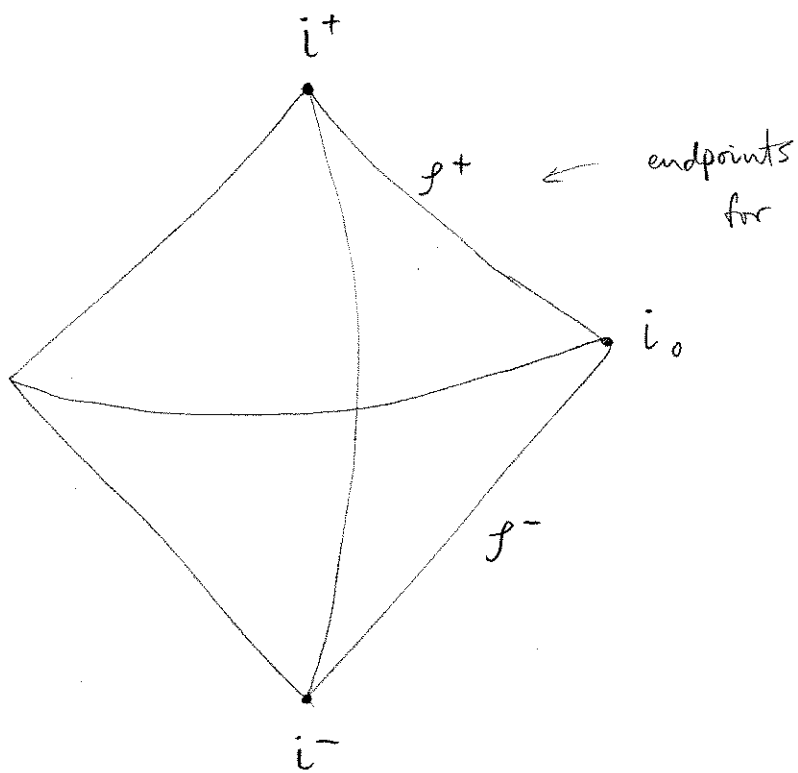


$\uparrow \begin{cases} u = t-r \\ v = t+r \end{cases}$

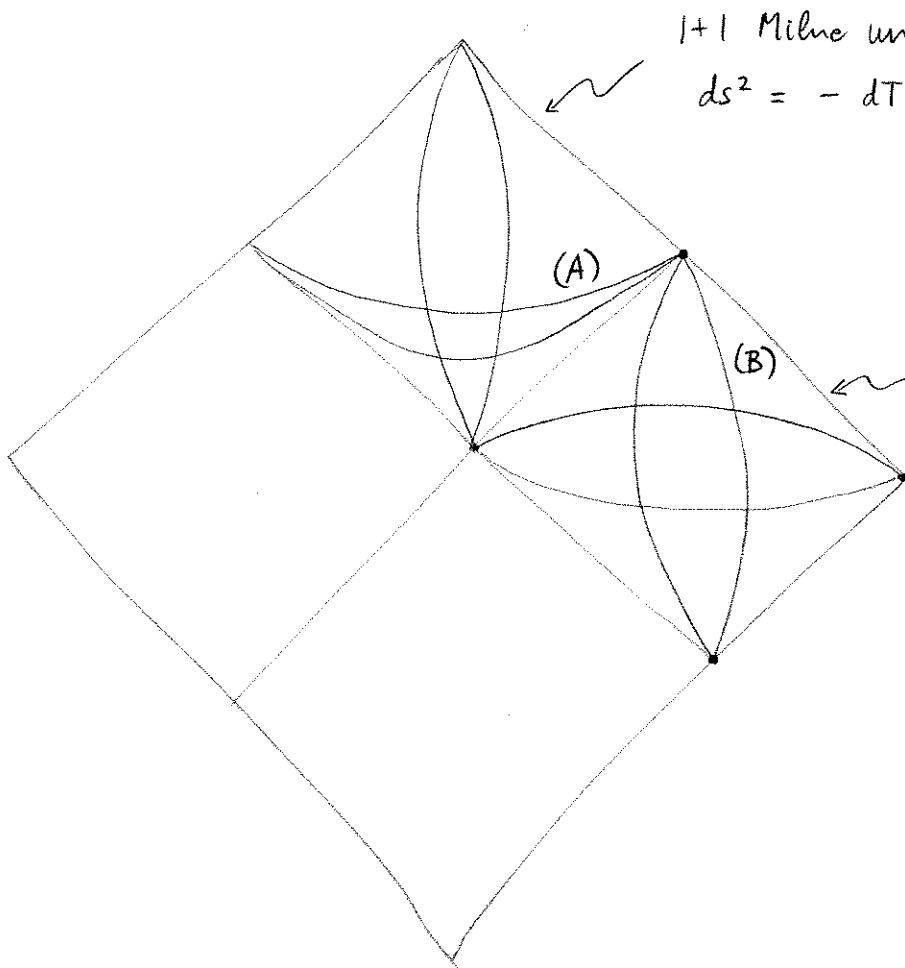
$\uparrow \begin{cases} u = \tan U \\ v = \tan V \end{cases}$

$-\frac{\pi}{2} < U < \frac{\pi}{2}$
 $-\frac{\pi}{2} < V < \frac{\pi}{2}$

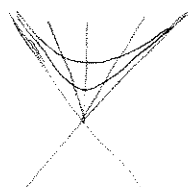
$= \sec^2(\tilde{t}-\tilde{r}) \sec^2(\tilde{t}+\tilde{r}) \cdot (-d\tilde{t}^2 + d\tilde{x}^2)$
 $\uparrow \begin{cases} u = \tilde{t}-\tilde{r} \\ v = \tilde{t}+\tilde{r} \end{cases}$



- null (geodesics) curves
- timelike traject. with asympt. velocity c (A)
- asymptotically null spatial curves (B)



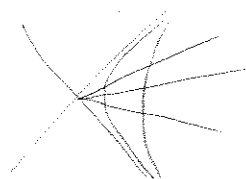
1+1 Milne universe
 $ds^2 = -dT^2 + T^2 d\chi^2$



1+1 Rindler wedge
 $ds^2 = -\xi^2 dT^2 + d\xi^2$

$\xi = ct \text{ (at.)} \rightarrow$ unif. accel. observer

rigid motion
 (different accel. for different values of ξ)



2- Examples in 4-D : Minkowski & Spherically symmetric BH's

* 3+1 Minkowski

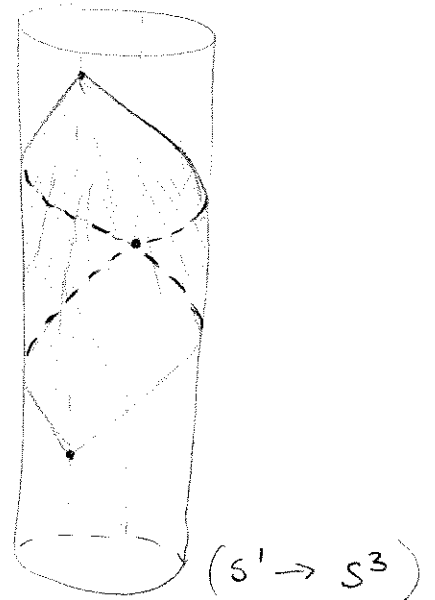
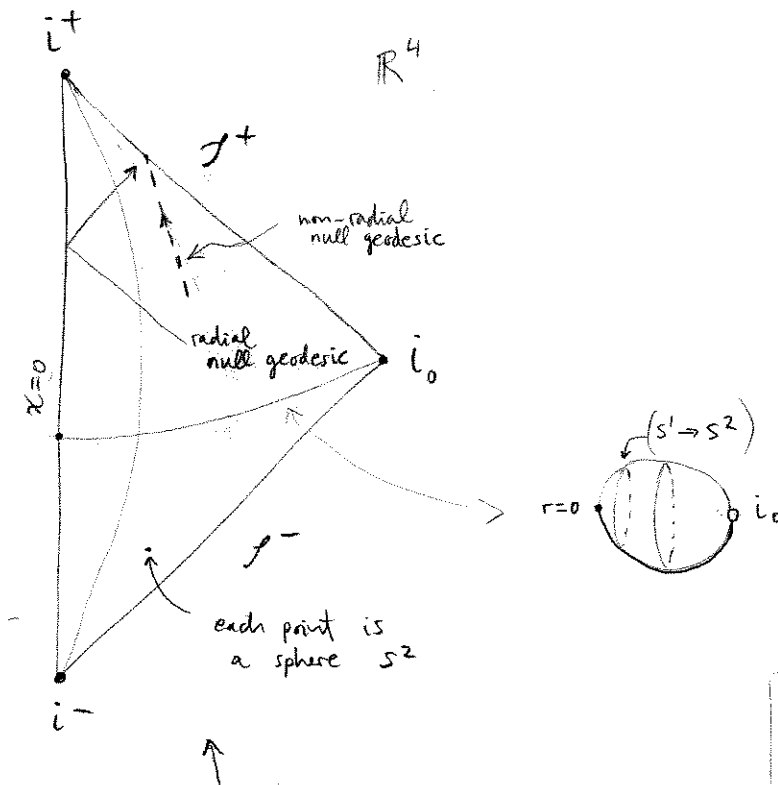
$$ds^2 = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) = -dudv + \underbrace{\left(\frac{v-u}{2}\right)^2}_{d\Omega^2} =$$

$$= \frac{1}{4} \sec^2 u \sec^2 v \left[-4 du dv + \sin^2(u-v) d\Omega^2 \right] =$$

$$= \sec^2(\tilde{t}-\chi) \sec^2(\tilde{t}+\chi) \left[-d\tilde{t}^2 + d\chi^2 + \sin^2\chi d\Omega^2 \right]$$

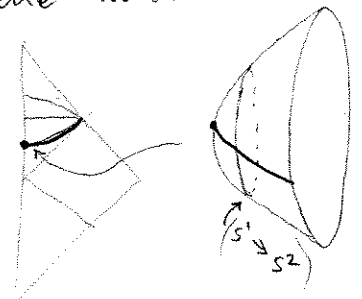
$$\begin{pmatrix} \chi = u-v \\ \tilde{t} = u+v \end{pmatrix}$$

Conformal to the Einstein static universe



- This structure is roughly used to define asymptotic flatness (asymptotically flat regions)

3+1 Milne universe



* 3+1 BH's (Spherically symmetric)

- Schwarzschild: $ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2$

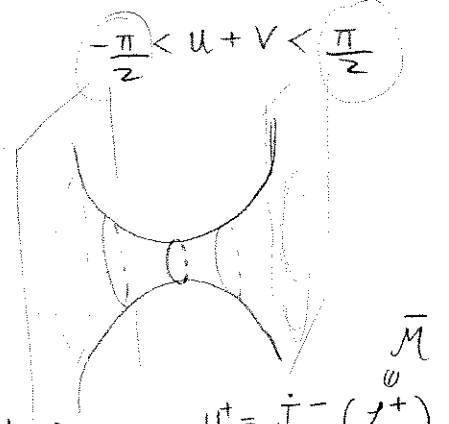
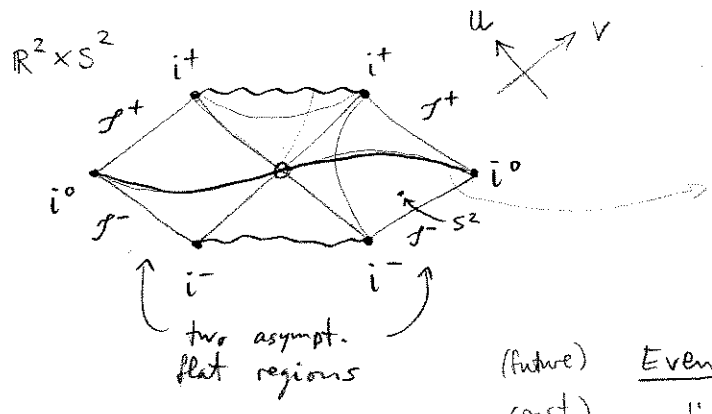
Kruskal maximal extension $\rightarrow ds^2 = F^2(t', r') (-dt'^2 + dr'^2) + r^2(t', r') d\Omega^2$

$F^2 = \frac{16M^2}{r} e^{-r/2M}$; $t'^2 - r'^2 = -(r-2M) e^{r/2M}$

$\begin{pmatrix} v' = t' + r' \\ u' = t' - r' \end{pmatrix} \rightarrow \begin{pmatrix} \tan V = \frac{v'}{\sqrt{2M}} \\ \tan U = \frac{u'}{\sqrt{2M}} \end{pmatrix}$

$-\frac{\pi}{2} < V < \frac{\pi}{2}$, $-\frac{\pi}{2} < U < \frac{\pi}{2}$

$-\frac{\pi}{2} < U + V < \frac{\pi}{2}$



(future) Event horizon = $H^+ = j^- (j^+)$
 (past) " " = $H^- = j^+ (j^-)$

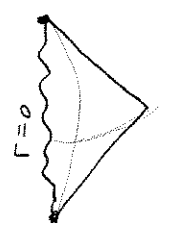
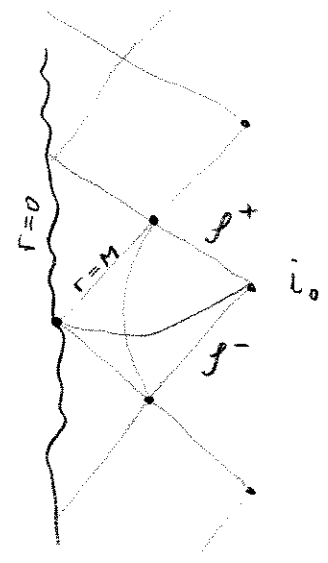
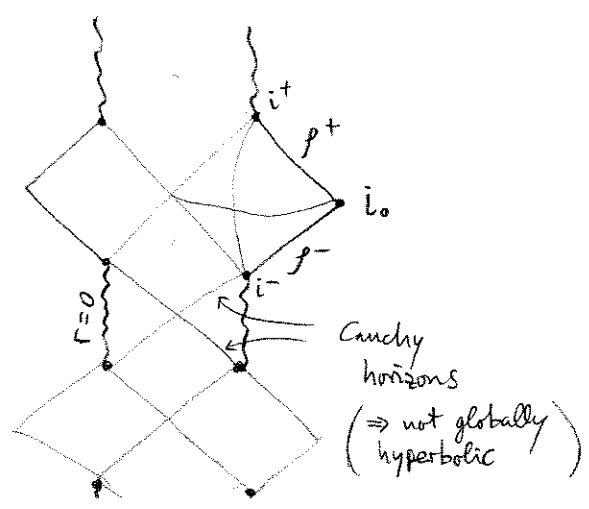
• Cauchy development & global hyperbolicity

- Reissner-Nordstrom: $ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dr^2 + r^2 d\Omega^2$

$Q^2 < M^2$

$Q^2 = M^2$
extremal BH

$Q^2 > M^2$
elem. particle



* Cosmological examples (spatially homogeneous & isotropic)

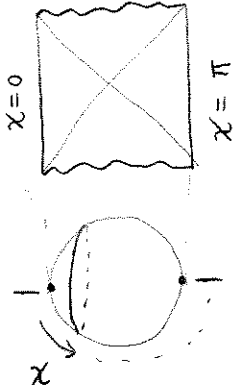
- General R-W metrics :

$k=1$ $ds^2 = a^2(\eta) [-d\eta^2 + d\chi^2 + \sin^2\chi d\Omega^2]$

exs. (i) radiation dominated
 $a \propto \sin \eta$

(ii) matter dominated
 $a \propto \sin^2\left(\frac{\eta}{2}\right)$

$R \times S^3$

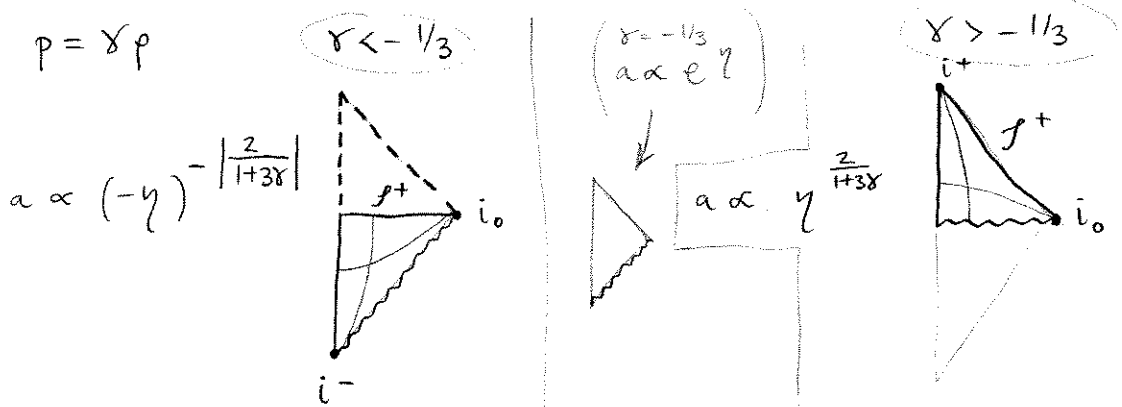


$k=0$

$ds^2 = a^2(\eta) [-d\eta^2 + d\tau^2 + \tau^2 d\Omega^2]$

exs. $p = \gamma \rho$

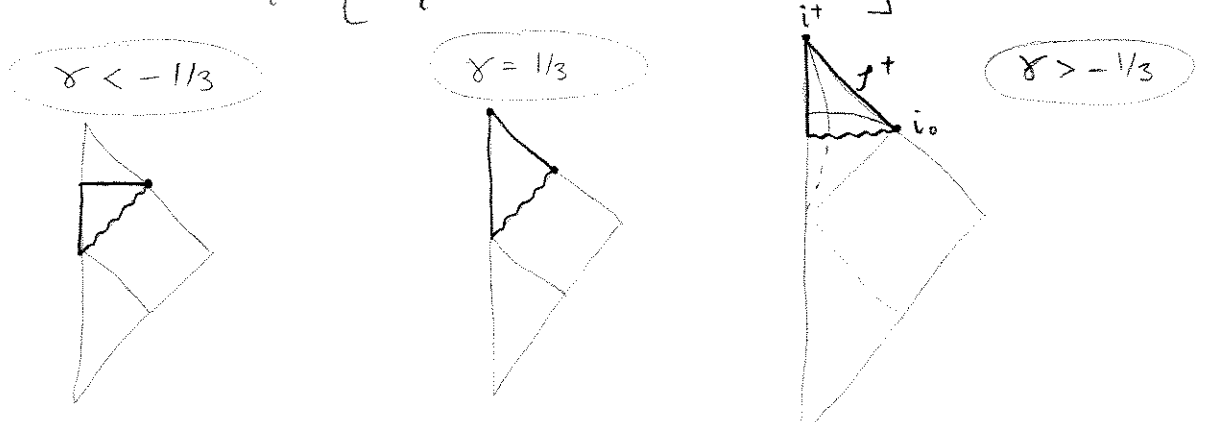
R^4



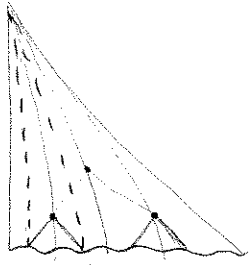
$k=-1$

$ds^2 = a^2(\eta) [-d\eta^2 + d\chi^2 + \text{sh}^2\chi d\Omega^2]$

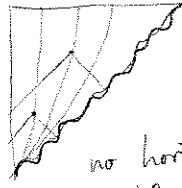
R^4



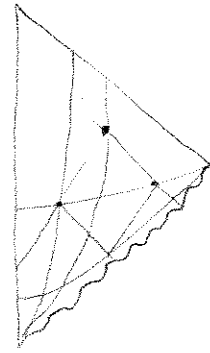
Particle horizon (horizon problem) :



particle horizon → horizon problem

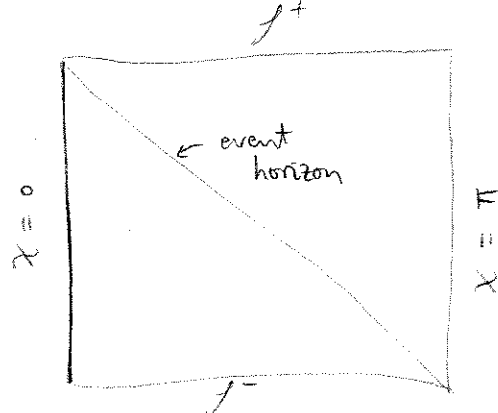
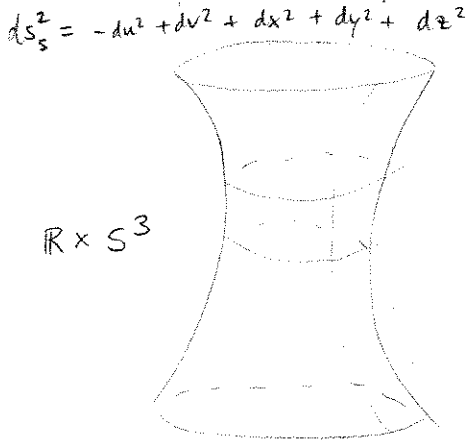


no horizon problem?



• de Sitter ($\Lambda > 0$, maximally symmetric)

$-u^2 + v^2 + x^2 + y^2 + z^2 = H^{-2} = \frac{\Lambda}{3}$ ← embedding in 4+1 Mink.
 (Geometric units $8\pi G = c = 1$)

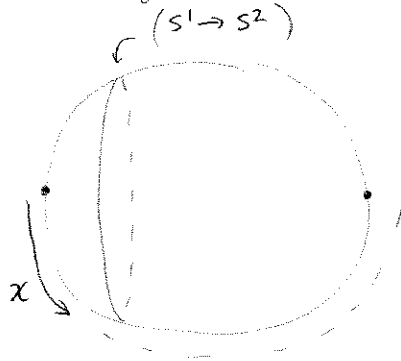
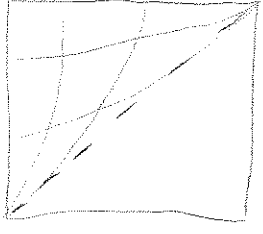


$$ds^2 = \left(\frac{1}{H \sin \chi}\right)^2 \left[-d\chi^2 + d\Omega^2 + \sin^2 \chi d\Omega^2\right]$$

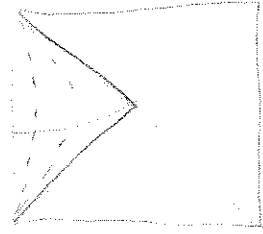
Other coord. systems =

$$ds^2 = \left(\frac{1}{H\eta}\right)^2 \left[-d\eta^2 + dr^2 + r^2 d\Omega^2\right]$$

(flat)



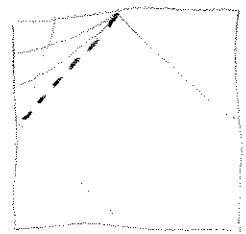
$$ds^2 = -(1-r^2H^2) dt^2 + \frac{dr^2}{(1-r^2H^2)} + r^2 d\Omega^2$$



(static)

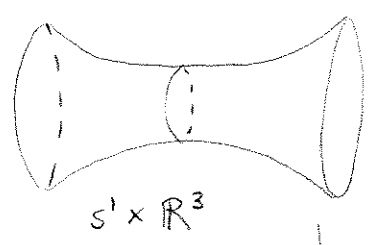
$$ds^2 = -dt^2 + \frac{1}{H^2} \text{sh}^2 Ht \left(d\chi^2 + \text{sh}^2 \chi d\Omega^2\right)$$

(open)

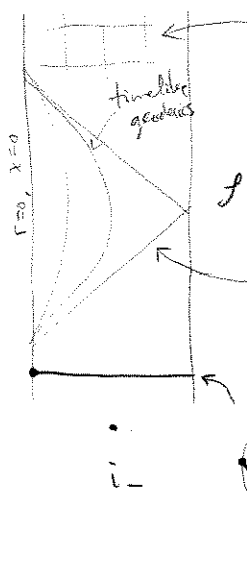


• Anti-de Sitter ($\Lambda < 0$, maximally symmetric)

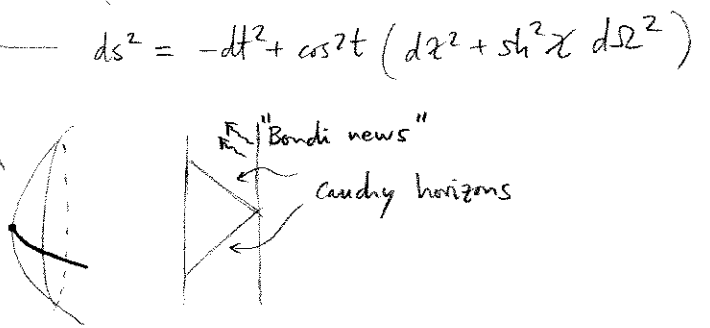
$-u^2 - v^2 + x^2 + y^2 + z^2 = -H^2 = \frac{\Lambda}{3}$ ← embedding in 3+2 flat space
 $ds^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2$. i^+
 $ds^2 = -(1 + \frac{|\Lambda|}{3} r^2) dt^2 + (1 + \frac{|\Lambda|}{3} r^2)^{-1} dr^2 + r^2 d\Omega^2$



universal covering space
(simply connected)
 \mathbb{R}^4



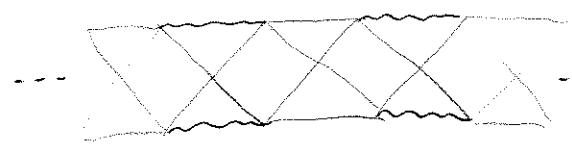
$ds^2 = -dr^2 \tilde{t}^2 + d\tilde{r}^2 + \text{sh}^2 \tilde{r} d\Omega^2$
 Alternatively,
 $ds^2 = -dt^2 + \cos^2 t (d\chi^2 + \text{sh}^2 \chi d\Omega^2)$



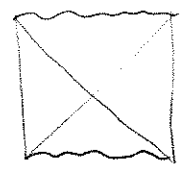
* Further examples:

• Schwarzschild - dS

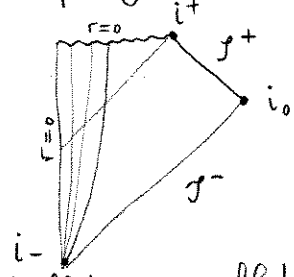
• Schwarzschild - adS



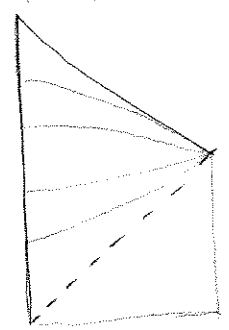
$ds^2 = -(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2) dt^2 + (1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2)^{-1} dr^2 + r^2 d\Omega^2$



• Collapsing b.h.



• Inflation + flat FRW ($\Lambda > 0$) ($\gamma > -1/3$)



• Evaporating b.h.

