

- 1. Conformal invariance of null geodesics:** Two metrics related by an overall scalar multiple function are said to be “conformally related”, or related by a “Weyl rescaling” or “Weyl transformation”. The light cones of two such metrics  $g_{\mu\nu}$  and  $A^2(x)g_{\mu\nu}$  are obviously the same, and hence so are the null curves. Show that in fact the null geodesic curves are also the same, but that the affine parameters on those curves are not the same. [For information on the non-affinely parametrized geodesic equation see the supplements link at the course web page.]

[Nordström’s 1913 theory of gravity was shown by Einstein and Fokker in 1914 to be interpretable as a curved metric theory. The metric in this theory is restricted to be *conformally flat*, i.e. of the form  $g_{\mu\nu}(x) = A^2(x)\eta_{\mu\nu}$ , where the single scalar function  $A(x)$  is the only field variable describing gravity. The result of this problem shows that in Nordström’s theory light is not deflected in a gravitational field, which contradicts observations. For information on this theory see for example “MTW”, i.e., *Gravitation* by Misner, Thorne, and Wheeler, or [wikipedia.org/wiki/Nordstrom's\\_theory\\_of\\_gravitation](http://wikipedia.org/wiki/Nordstrom's_theory_of_gravitation)]

- 2. Affine parameter on black hole horizon generators:** Consider the Eddington-Finkelstein (EF) line element (see hw2).
- (a) Show that the radial coordinate  $r$  is an affine parameter along both ingoing and outgoing null geodesics (light rays), except for the outgoing one that sits on the horizon.
  - (b) The null geodesics on the horizon are called “horizon generators”. Show that the (“advanced time”) coordinate  $v$  is related to the affine parameter  $\lambda$  on the horizon generators by  $d^2v/d\lambda^2 = -\kappa(dv/d\lambda)^2$ , where  $\lambda$  is an affine parameter and  $\kappa = 1/4M$  is the “surface gravity” of the black hole. This means that  $v$  is not an affine parameter along the horizon generators.
  - (c) Show that, along the horizon generators,  $\exp(\kappa v) = a\lambda + b$ , where  $a$  and  $b$  are constants. Thus as  $v$  goes to negative infinity, the affine parameter  $\lambda$  covers only a finite range. This means that the EF coordinate patch does not cover the whole spacetime. We’ll see later what’s missing. Whatever it is, it is not relevant in a situation where the black hole formed at some finite time in the past from gravitational collapse, since the spacetime inside the collapsing stuff is not described by the EF line element.

- 3. Christoffel symbol:** In class we wrote the geodesic equation in the form

$$\frac{d}{d\lambda}(g_{\alpha\nu}\dot{x}^\nu) - \frac{1}{2}g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu = 0, \quad (1)$$

where the overdots signify derivative with respect to an affine parameter  $\lambda$ , e.g. the proper time in the case of timelike geodesics. Show that this equation is equivalent to the equation

$$\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0, \quad (2)$$

where  $\Gamma^\alpha_{\mu\nu}$  is the *Christoffel symbol(s)*, defined by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}). \quad (3)$$

Here  $g^{\alpha\beta}$  is the *inverse metric*, which satisfies  $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$  with  $\delta^\alpha_\gamma$  the Kronecker delta.