

## Phys 675

### Introduction to Relativity, Gravitation and Cosmology

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## Notes

Please don't assume these notes are complete. I will try to indicate here at least roughly what was covered, and will sometimes include additional material. There are fairly complete notes from previous years, linked at the course web page.

### Tuesday, Sept. 2

+ Introduced the class, the syllabus, the plan, course webpage, Piazza.

+ **Newtonian mechanics** of a particle: The Lagrangian is  $L = \frac{1}{2}m(d\vec{x}/dt)^2$ . This depends upon certain elements of **Newtonian spacetime structure** to be meaningful:

1.  $dt$ : an absolute time function, defined up to a constant shift.
2. inertial structure: rule for identification of constant  $t$  slices up to a uniform velocity
3. Euclidean geometry on each constant  $t$  slice

+ **Relativistic spacetime structure**: everything we need comes from the **line element**  $ds^2$  also called the **invariant interval**. In "Cartesian" coordinates this takes the form

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \quad (1)$$

The displacements  $(dt, dx, dy, dz)$  with  $ds^2 = 0$  make a double cone, the **light cone**, dividing the neighborhood of a point into the **future timelike, past timelike, and spacelike displacements**. The displacements along the cone are **lightlike**.  $\sqrt{-ds^2}$  gives  $c$  times the time along a timelike displacement, also called the **proper time**, and  $\sqrt{ds^2}$  gives the spatial length of a spacelike displacement. Lightlike displacements have  $ds^2 = 0$  so are also called **null**: No time or distance elapses along the light cone.

+ **Relativistic mechanics**: The action for a nonrelativistic particle is  $\int L dt$ . What can replace this for a relativistic particle? The only thing we can construct with the available structure, i.e. with the line element, is  $\int \sqrt{-ds^2}$ . This has dimensions of length, and doesn't refer to the particle mass, so can't be right. Both

of these flaws are fixed if we multiply by  $mc$ , and in fact a minus sign is needed. So the action for a relativistic particle is

$$S = -mc \int \sqrt{-ds^2}.$$

This works: make a Taylor expansion in  $v^2/c^2$ :

$$\sqrt{-ds^2} = \sqrt{c^2 dt^2 - dx^2} = c dt \sqrt{1 - v^2/c^2} = c dt \left(1 - \frac{1}{2}v^2/c^2 - \frac{1}{8}v^4/c^4 - \dots\right)$$

Then the action becomes  $\int dt(-mc^2 + \frac{1}{2}mv^2 + \frac{1}{8}mv^4/c^2 + \dots)$ . The second term is the nonrelativistic action. The first arises from a constant potential energy of the particle, called the **rest energy**, or **rest mass**.

#### Thursday, Sept. 4

+ The relativistic Lagrangian for a massive particle, written in a fixed inertial frame, is  $L = -mc^2(1 - v^2/c^2)^{1/2}$ . The momentum conjugate to the spatial coordinates  $x^i$  is defined as usual as  $p_i = \partial L / \partial \dot{x}^i$ , where the dot represents  $d/dt$ . The Hamiltonian is  $H = p_i \dot{x}^i - L$ , which is also the total energy  $E$ . (No inner product is involved in the expression  $p_i \dot{x}^i$ .) In terms of the velocity  $v^i = dx^i/dt$  and the **gamma factor**  $\gamma = (1 - v^2/c^2)^{-1/2}$  we find:

$$p^i = \gamma m v^i, \quad E = \gamma m c^2.$$

Given the expansion  $\gamma = 1 + \frac{1}{2}v^2/c^2 + \frac{1}{8}v^4/c^2$  we see that  $p^i$  reduces to  $mv^i$  in the non relativistic limit and  $E$  becomes  $mc^2 + \frac{1}{2}mv^2$ , so this energy includes the rest energy. Useful relations between these quantities are  $p^i = (E/c^2)v^i$  and

$$E^2/c^2 - p^i p^i = m^2 c^2,$$

with a sum over the repeated spatial index  $i$ . This last is called the **mass shell** condition, since it is the equation of a hyperboloidal shell in the  $E$ - $p^i$  space.

+ **Massless particles** ( $m = 0$ ) can have nonzero energy and momentum if the product  $\gamma m$  remains nonzero, which requires that  $\gamma = \infty$ , which requires that  $v^2 = c^2$ , that is, the particle must move at the speed of light. The mass shell condition has a regular limit:  $E^2/c^2 = p^i p^i$ , which is the equation of a *cone*, rather than a “shell”, in the  $E$ - $p^i$  space.

+ The **4-velocity**  $u^\alpha$  of the particle is defined as the infinitesimal spacetime displacement  $dx^\alpha = (c dt, dx^i)$  divided by the proper time  $d\tau = (-ds^2)^{1/2}/c$

along that displacement. The relation between proper time and coordinate time is  $d\tau = dt/\gamma$ , hence

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \gamma(c, v^i).$$

We can write the momentum and energy in terms of the 4-velocity as  $p^i = mu^i$  and  $E/c = mu^0$ , using the superscript 0 for the time component. The **4-momentum** is defined by

$$p^\alpha = mu^\alpha,$$

and has the components  $(E/c, p^i)$ .

+ *From here on, let us choose units such that  $c = 1$ .*

+ The **Minkowski inner product** is defined by the expression for the invariant interval,  $ds^2 = -dt^2 + dx^i dx^i = (dt, dx^i) \cdot (dt, dx^i)$ . It is like the Euclidean inner product in three dimensional space, but the time term comes with a minus sign. In terms of this inner product, the 4-velocity is a unit vector, i.e.  $u \cdot u = -1$ . We write the inner product of a 4-vector with itself as the “square”. Thus we have  $u^2 = -1$ , and the 4-momentum satisfies  $p^2 = -m^2$ . This is equivalent to the mass-shell condition. Just as the interval  $ds^2 = (dx^i)^2$  is invariant, i.e. the same in all inertial coordinate systems (this will be explained more fully below), so is the square of the 4-velocity and the square of the 4-momentum.

+ **Opacity of the universe to gamma rays above a certain energy:** The pair production process  $\gamma\gamma \rightarrow e^+e^-$  absorbs gamma ray photons if they collide with other photons with sufficient energy. The **threshold** for such a reaction is defined as the minimum energy needed, given a head on collision with a target photon of a given energy. The target photons are from the background light of the universe. They could be microwave photons from the primordial radiation, but it turns out that the actual cutoff for gamma rays from distant sources comes from the far infrared background generated by stars.

The 4-momenta of gammas 1 and 2 satisfy  $p_{1,2}^2 = 0$ , while those of the pair satisfy  $p_\pm^2 = -m_e^2$ , where  $m_e$  is the electron mass. Energy and momentum conservation imply equality of the initial and final total 4-momentum,  $p_1 + p_2 = p_+ + p_-$ . We can obtain from this a scalar equation by squaring both sides, which will be enough information since we seek only one quantity, the threshold energy. This yields  $2p_1 \cdot p_2 = 2p_+ \cdot p_- - 2m_e^2$ . At threshold the electron and positron should have the minimum energy, so they should be created at rest in the zero momentum frame (ZMF). This means that  $p_+ = p_-$ , so  $p_+ \cdot p_- = p_+^2 = -m_e^2$ . Thus we

have  $p_1 \cdot p_2 = -2m_e^2$ . Now  $p_1 = (E_1, E_1, 0, 0)$  and  $p_2 = (E_2, -E_2, 0, 0)$ , so  $p_1 \cdot p_2 = -2E_1E_2$ , hence  $E_1E_2 = m_e^2$ .

Actually there is a much simpler way to arrive at the same conclusion: start by analyzing the collision in the ZMF. Then the photons have equal energy, and by energy conservation each must be equal to the electron rest mass, so  $E_1E_2 = m_e^2$ . But the final calculation above shows that in any frame in which the photons collide head-on we have  $E_1E_2 = -p_1 \cdot p_2$ . Since the right hand side is an invariant, evidently the product of the energies is the same in all those frames. So the threshold condition  $E_1E_2 = m_e^2$  applies in all those frames.

For a microwave background photon  $E_2 \sim 10^{-4}$  eV, and  $m_e = 0.511$  MeV so we obtain  $E_1 \sim 10^{15}$  eV. But for a far infrared photon of  $E_2 \sim 10^{-2}$  eV we find the lower threshold  $\sim 10^{13}$  eV = 10 TeV.

## Tuesday, Sept. 9

+ **Wave 4-vector:**  $k = (\omega, \vec{k})$ . For a photon, the 4-momentum is  $p = \hbar k$ .

+ **Doppler shift for light:** If a source moving with coordinate velocity  $v$  relative to an observer emits light with proper frequency  $\omega_S$ , at an angle  $\theta$  from the direction of motion, what is the observed frequency  $\omega_O$ ? This is derived in Hartle's book (pp. 92-93) by using a Lorentz boost to relate the components of the wave four-vector  $k$  in the rest frame of the source to those in the observer's rest frame. It can also be done without making any Lorentz transformation, using invariants, as follows.

The frequency observed by an observer with four-velocity  $u$  is given by the scalar

$$\omega_u = -k \cdot u. \quad (2)$$

The inner product on the right hand side is an invariant. ("Frequency" is the time component of a 4-vector, so is not invariant. But the frequency measured by a specified observer is invariant.) We're given  $\omega_S$  in the source frame, and  $v$  and  $\theta$  in the observer frame, so let's use (2) to write  $\omega_S = -k \cdot u_S$ , and evaluate the right hand side using components in the observer frame. These components are  $k = \omega_O(1, \cos \theta, \sin \theta, 0)$  and  $u_S = \gamma(1, v, 0, 0)$ , so we obtain  $\omega_S = \omega_O \gamma(1 - v \cos \theta)$ , or  $\omega_O = \omega_S / \gamma(1 - v \cos \theta)$ . For  $\theta = 0, \pi$  this reduces to  $\sqrt{(1 \pm v)/(1 \mp v)}$ , while for  $\theta = 0$  it's the "transverse Doppler shift",  $\omega_S = \omega_O \gamma$ , which is nothing but the relativistic time dilation effect: in terms of the phase  $\phi$ , we have  $\omega_S = d\phi/dt_S = (d\phi/dt_O)(dt_O/dt_S) = \omega_O \gamma$ .

+ **Geometric derivation of the spacetime interval:** Using the basic postulates of special relativity, 1) equivalence of inertial observers, 2) light propagates

in a way that is independent of the motion of the source, and 3) spacetime has the same properties at all events (it is homogeneous and isotropic), one can give a simple ‘geometric’ derivation of the invariant interval. Part of this derivation amounts to a *definition* of the coordinates that a given observer constructs in spacetime, using radar and proper time measurements. See my Phys 410 notes for details.

+ **Spacetime interval and Euclidean area:** in two-dimensional spacetime diagrams, the proper time along a timelike line segment is proportional to the Euclidean distance along the segment in the diagram, but the proportionality factor depends on the timelike line. However, the *square* the proper time is proportional to the Euclidean area of the “light rectangle” that forms the causal domain of the line segment, with a *constant* proportionality factor. Thus one can compare proper times by comparing corresponding Euclidean areas in the diagram. This fact is derived geometrically from the postulates of relativity in the article *Spacetime and Euclidean Geometry*, by Dieter Brill and Ted Jacobson, (<http://arxiv.org/abs/gr-qc/0407022>). Geometric derivations of the time dilation effect and the spacetime Pythagoras theorem (interval) are also given there.

## Thursday, Sept. 11

+ **Relativistic beaming:** In a homework problem it is shown that a source of electromagnetic radiation with isotropic proper specific intensity  $I_{\omega_*}$ , when moving at coordinate velocity  $v$  relative to an observer, has observed specific intensity  $I_{\omega} = dE/d\omega dt d\Omega$  given by  $I_{\omega}/I_{\omega_*} = [\gamma(1 - v \cos \theta)]^{-1/3} \approx [2\gamma/(1 + \gamma^2 \theta^2)]^3$ , where the approximation is good when the angle is small compared to unity. Thus within the cone  $\theta < 1/\gamma$  the intensity scales as  $\gamma^3$ , whereas outside this cone it scales as  $\gamma^{-3}$ .

The beaming can be understood intuitively using a spacetime diagram. An emitter worldline and two back-to-back emitted photons form a two dimensional plane in spacetime. In the frame of the emitter, this plane intersects the light cone vertically in two light rays separated by the angle  $\pi$ , which go through the vertex of the cone. In the frame of another observer for whom the emitter is moving with coordinate speed  $v$  in the  $x$  direction, this plane still goes through the vertex, but is tilted toward the  $x$  direction, and thus intersects the cone in two light rays separated by a smaller angle  $\theta$ . As  $v \rightarrow 1$ , this angle goes to zero. In an observer time  $t$ , the emitter and the two back-to-back photons all acquire the  $x$  coordinate  $vt$ , and the photons travel a distance  $ct$  at angle  $\theta$ , so  $\cos \theta = v/c$ , and thus  $\sin \theta = 1/\gamma$ . For small angles this yields  $\theta \approx 1/\gamma$ .

+ **Minkowski coordinates and Lorentz transformations:** Coordinates for which the line element takes the form (1) are generally called **Minkowski coordinates**. They could also reasonably be called “spacetime Cartesian coordinates”. How wide is the class of Minkowski coordinate systems? A constant shift of the coordinates will not change the form of the line element (1), nor will a rotation of the spatial Cartesian coordinates. The only additional type of coordinate change for which the line element will retain this form is called a Lorentz boost.

To establish this fact we employ a convenient notation that will also be used in general relativity. Group the coordinates into a single indexed coordinate, viz.  $\{x^\alpha\} = (t, x, y, z)$ . Then the line element reads

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta, \quad \eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1) \quad (3)$$

The array  $\eta_{\alpha\beta}$  is called the **Minkowski metric**. Now suppose the coordinates  $x^\alpha$  are functions of some other set of four coordinates  $y^\alpha$ . Then

$$dx^\alpha = \frac{\partial x^\alpha}{\partial y^\mu} dy^\mu \quad (4)$$

where summation over the repeated index  $\mu$  is implicit. This is the **Einstein summation convention**. Thus

$$ds^2 = \left( \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} \eta_{\alpha\beta} \right) dy^\mu dy^\nu. \quad (5)$$

The expression in the parenthesis is the metric tensor for the Minkowski metric, expressed in the  $y^\alpha$  coordinate system, which is a Minkowski coordinate system if and only if

$$\frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} \eta_{\alpha\beta} = \eta_{\mu\nu}. \quad (6)$$

This is 10 conditions at every point of spacetime. It actually implies that  $x^\alpha(y)$  are linear functions<sup>1</sup>,

$$x^\alpha = \Lambda^\alpha{}_\mu y^\mu + b^\alpha, \quad (9)$$

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<sup>1</sup>I think one can show this by using the fact that straight lines  $x^\alpha = u^\alpha \tau + x_0^\alpha$  must all map to straight lines  $y^\alpha = v^\alpha \lambda + y_0^\alpha$ , where  $\lambda$  is a parameter and  $u^\alpha$ ,  $x_0^\alpha$ ,  $v^\alpha$  and  $y_0^\alpha$  are all constants, which follows from the fact that these lines are determined by the intrinsic geometry of the Minkowski metric. It can also be seen by taking the derivative of (6) with respect to  $y^\sigma$ , which yields

$$\left( \frac{\partial^2 x^\alpha}{\partial y^\sigma \partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} + \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial^2 x^\beta}{\partial y^\sigma \partial y^\nu} \right) \eta_{\alpha\beta} = 0. \quad (7)$$

One way to satisfy this is if the second derivatives all vanish, but is that the only way? Eq. (7) has free indices  $\sigma\mu\nu$  and is symmetric in  $\mu\nu$ , so it represents  $4 \times 10$  equations. On the other hand, the second derivatives  $\partial^2 x^\alpha / \partial y^\sigma \partial y^\mu$  are symmetric in  $\sigma\mu$ , so represent  $4 \times 10$  independent components at each spacetime point. This suggests that the conditions (7) uniquely determine the second derivatives,

where  $\Lambda^\alpha{}_\mu$  and  $b^\alpha$  are constants. The  $b^\alpha$  are spacetime translations. The  $\Lambda^\alpha{}_\mu$  must satisfy (6), i.e.

$$\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} = \eta_{\mu\nu}. \quad (10)$$

This is 10 equations on the 16 independent components of  $\Lambda^\alpha{}_\mu$ , so one expects a  $16 - 10 = 6$  parameter family of solutions, which form a group called the **Lorentz group**. Three of these parameters correspond to rotations. The other three correspond to **Lorentz boosts**. Taken together also with the spacetime translations, these form the **Poincaré group**. Considering (10) as a matrix equation  $\Lambda^T \eta \Lambda = \eta$ , we can take the determinant of both sides to see that  $(\det \Lambda)^2 = 1$ , so in particular  $\Lambda$  must be invertible (which is in any case required if the coordinate transformation is to be invertible and smooth).

To find the form of a Lorentz boost, let's restrict to the case where the coordinates  $y$  and  $z$  are unchanged. Then we can think of  $\Lambda$  as a  $2 \times 2$  matrix. We may immediately write down four **discrete symmetry** solutions,  $\Lambda = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ , where the two  $\pm$  signs are independent. These correspond to the identity,  $t$  reflection,  $x$  reflection, and combined  $t$  and  $x$  reflection. The two single reflections have determinant  $-1$ , so the general solution may be found by first finding all solutions with  $\det \Lambda = 1$ , and then multiplying all of those by the time reflection. It is easy to show that unit determinant matrices satisfying  $\Lambda^T \eta \Lambda = \eta$  must have the form  $\Lambda = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ , with  $a^2 - b^2 = 1$ . We may restrict to  $a > 0$  if we include afterward also those  $\Lambda$  obtained by multiplication by minus the identity, which is combined  $t, x$  inversion. Thus we can parametrize the general two-dimensional Lorentz transformation as  $\Lambda = \begin{pmatrix} \cosh \psi & \sinh \psi \\ \sinh \psi & \cosh \psi \end{pmatrix}$  times one of the four discrete symmetry solutions. The hyperbolic angle  $\psi$  is called the **rapidity**. In terms of the coordinate velocity  $v$  of the boost, we have  $v = \tanh \psi$ , and  $\gamma = \cosh \psi$ , so the Lorentz boost may also be written as  $\Lambda = \gamma \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}$ .

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and thus that zero is the only solution. In fact this is the case. To show it requires a bit of fancy footwork: write the equation three times, with the free indices relabeled cyclically, and then combine the equations for example as  $\sigma\mu\nu + \mu\nu\sigma - \nu\sigma\mu$ . Using the symmetry of the mixed partials, one then sees that all terms cancel except two which are identical, and one is left with

$$\frac{\partial^2 x^\alpha}{\partial y^\sigma \partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} \eta_{\alpha\beta} = 0. \quad (8)$$

As explained in the text, the Jacobian must be invertible, and  $\eta_{\alpha\beta}$  is invertible as a matrix, so the factor  $\frac{\partial x^\beta}{\partial y^\nu} \eta_{\alpha\beta}$  is invertible. We may thus multiply (8) by its inverse and conclude that the second partials all vanish.

+ **Inertial structure of Minkowski spacetime:** As emphasized in the first lecture of the semester, the invariant interval  $ds^2$  encodes all of the structure needed to formulate mechanics, whereas Newtonian mechanics requires a separate absolute time interval, spatial geometry, and inertial structure. It's clear how the interval encodes time and length structure, but how does it encode inertial structure? *Inertial trajectories in spacetime are straight lines in Minkowski coordinates.* For this reason, these are also sometimes called **inertial coordinates**. For this to make sense it is necessary that all Minkowski coordinate systems determine the same inertial trajectories. This is the case, since the relation between such coordinates is linear. A more intrinsic way to determine the inertial trajectories is to identify them as the spacetime paths that *maximize the proper time between fixed endpoints.*

+ **The key idea of General Relativity:** Einstein focused on the extremely well known but utterly mysterious fact that, in Newtonian gravity, the gravitational force is proportional to the inertial mass of the object it is acting on:  $\mathbf{F}_{\text{grav}} = m\mathbf{g}$ , where  $\mathbf{g}(x, t)$  is the gravitational field. If the only force is gravitational, the acceleration of the body is thus  $\mathbf{a} = \mathbf{g}$ , independent of the mass. So all bodies fall in the same way. Moreover, even when other forces are acting, the effects of gravity can be locally removed by adopting a “freely falling” reference frame with acceleration  $\mathbf{g}$  relative to what Newton would consider an inertial frame. Einstein proposed that we should think of it the other way around: the freely falling frame is the inertial one. So, for example, sitting in my chair, I am in a frame accelerating upwards relative to Einstein's **local inertial frame**.

Another way to say this is that Einstein interprets the gravitational force as an “inertial force” (a.k.a. a fictitious force or pseudo-force) analogous to a centrifugal or Coriolis force, arising in a reference frame with acceleration. Newton's 2nd law  $\mathbf{F} = m\mathbf{a}$  holds in a (Newtonian) inertial frame, but can be expressed in a non-inertial frame. Using  $\mathbf{a} = \mathbf{a}_{\text{rel}} + \mathbf{a}_{\text{frame}}$ , where  $\mathbf{a}_{\text{rel}}$  is the acceleration relative to a non-inertial frame with acceleration  $\mathbf{a}_{\text{frame}}$ . Then the 2nd law can be expressed as

$$\mathbf{F} - m\mathbf{a}_{\text{frame}} = m\mathbf{a}_{\text{rel}}. \quad (11)$$

The inertial force  $-m\mathbf{a}_{\text{frame}}$  is recognized as such by its proportionality to the inertial mass. Einstein's idea is to change the definition of an inertial frame, and interpret the gravitational force  $m\mathbf{g}$  as an inertial force, brought about by using a reference frame with acceleration  $-\mathbf{g}$ .

Einstein's local inertial frames cannot be extended to global inertial frames, since  $\mathbf{y}$  is not constant. For example, at different points near the surface of the earth the free-fall frames are falling inward radially, and the radial direction depends on where you are, so that nearby freely falling particles have slightly different accelerations. You could recognize this in a falling elevator: if release a spherical cluster

of particles, as the cluster falls it will deform to an ellipsoid, compressed in the transverse direction and stretched in the falling direction. If it weren't for that, we could just cancel off gravity once and for all by changing the reference frame. *The true essence of gravity is this "tidal deformation"*, i.e. the variation of the local inertial frames.

+ **Spacetime curvature and the tidal field:** The inertial structure of spacetime is determined in special relativity by the line element, so a *spatially varying* inertial structure must be described by a spatially varying line element, that is, by a *deformation of the geometry of spacetime*. In fact, the curvature of the spacetime geometry captures the notion of varying inertial structure. For example, freely falling paths can start out parallel in spacetime, and be pulled together by the gravitational tidal field. That parallel lines do not remain parallel is a sign of curvature.

## Tuesday, Sept. 16

+ **Generic line element:** A general spacetime geometry is determined by a line element of the form

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \quad (12)$$

where the 10 independent components of the (symmetric) **metric tensor**  $g_{\mu\nu}$  are functions of the four coordinates. The meaning of  $ds^2$  is the same as in the flat spacetime of special relativity.

+ **Inertial motion in curved spacetime:** The motion of a test particle is determined by maximizing the proper time with respect to variations of the path that vanish outside a small interval. (There may exist variations that decrease the proper time, but they would not vanish outside a small interval.)

+ **Time runs faster higher up:** Otherwise, why should a tossed object go up and then back down to the same height? What determines the height are shape of the path is the tradeoff between going higher where time is running faster, and the extra time dilation due to the need to move faster. In the limit up and down at the speed of light, no proper time passes, so that's obviously too far!

+ **Matching to the Newtonian limit:** Suppose the metric is nearly flat, in the sense that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} \ll 1, \quad (13)$$

using coordinates  $x^\mu = (t, x^i)$ ,  $i = 1, 2, 3$ . The action of a particle of mass  $m$  is then

$$S = -m \int \sqrt{-ds^2} = -m \int dt \sqrt{1 - v^2 - h_{tt} - 2h_{ti}v^i - h_{ij}v^i v^j}, \quad (14)$$

where  $v^i = dx^i/dt$  is the coordinate velocity, and  $v^2 = \delta_{ij}v^i v^j$ . Now suppose the particle velocity is very small compared to the speed of light,  $v^i \ll 1$ . Then, assuming  $h_{ti}$  and  $h_{ij}$  are not much larger than  $h_{tt}$ , the last two terms under the square root are much smaller than  $h_{tt}$ . Neglecting these terms, and expanding the square root, we then have

$$S \approx \int dt \left( \frac{1}{2}mv^2 + \frac{1}{2}mh_{tt} - m \right). \quad (15)$$

This looks just like the Newtonian action for a particle in a gravitational potential per unit mass  $\Phi_N = -\frac{1}{2}h_{tt}$ , neglecting the constant mass term that doesn't affect the equation of motion. This yields the important relation between the metric perturbation and the Newtonian potential,

$$h_{tt} = -2\Phi_N. \quad (16)$$

At the surface of earth,  $\Phi_N/c^2 = GM_e/c^2 r_e = g_e r_e/c^2 = (10m/s^2)(6.3 \times 10^6)/(3 \times 10^8)^2 \sim 10^{-9}$ , so the metric perturbation is indeed very small compared to 1. (Note that I divided by  $c^2$  because I wanted to use SI units, and  $h_{tt}$  is dimensionless.) By the way, the change of  $h_{tt}$  is of order  $10^{-16}$  per meter. Atomic clocks are now just able to directly measure this sort of difference per meter.

**Geodesic equation:** To find the paths for which the action, or the proper time integral, is stationary, we assign an arbitrary parameter  $\lambda$  to the points along the path, which is then described by four functions  $x^\mu(\lambda)$ . Then

$$ds^2 = g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu (d\lambda)^2, \quad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}, \quad (17)$$

so the action can be written as

$$S = \int L d\lambda, \quad L = -m \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}. \quad (18)$$

The stationary condition  $\delta S = 0$  then corresponds to the Euler-Lagrange equations,

$$\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0. \quad (19)$$

The equations are simpler if we now choose the parameter  $\lambda$  such that  $\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$  is constant. This is the case if  $\lambda$  is linearly related to the proper time  $\tau$ , so let's use the proper time. Equations (19) then become

$$\frac{d}{d\tau}(g_{\mu\alpha}\dot{x}^\mu) - \frac{1}{2}g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu = 0, \quad (20)$$

where the comma notation indicates partial derivative, e.g.  $g_{\mu\nu,\alpha} = \partial g_{\mu\nu}/\partial x^\alpha$ .

## Thursday, Sept. 18

### + Extremal time versus longest time geodesics

+ **Local inertial coordinates at a point  $p$** :  $g_{\mu\nu}(p) = 0$  and  $g_{\mu\nu,\alpha}(p) = 0$ . Such coordinates always exist. In such a coordinate system, the geodesic eqn (20) at  $p$  reads simply  $(g_{\mu\alpha}\ddot{x}^\mu)(p) = 0$ . Since the metric is invertible, this is equivalent to  $\ddot{x}^\mu(p) = 0$ , implying that the curve is linear to second order in the parameter  $\lambda$ ,  $x^\mu = x^\mu(p) + \dot{x}^\mu(p)\lambda + O(\lambda^3)$ . So the geodesic equation can be understood as simply the statement that in local inertial coordinates the curve has instantaneously vanishing coordinate acceleration. Given any geodesic, it is always possible to choose a single coordinate system that is locally at all points on that geodesic. As a Riemannian example, consider the  $\theta = \pi/2$  equator of a 2-sphere.

+ **Schwarzschild metric (1916)**: Up to coordinate transformations this is the unique spherical, vacuum solution to Einstein's equation. "Vacuum" refers to empty spacetime, for example outside of a star or black hole, where the only field in Einstein's equation is the metric. Birkhoff's theorem shows that this is the only spherically symmetric solution; in particular, time independence of the solution is a consequence of spherical symmetry, since general relativity has no spherically symmetric wave mode.

+ **Orbits in the Schwarzschild spacetime**: The motion of a small body is actually determined by the Einstein field equation. To the extent that the self-gravity can be neglected, we can think of it as a test-body that moves on a geodesic of the background spacetime. If the background is the Schwarzschild spacetime, the world line of a small body can given by four functions in Schwarzschild coordinates,  $(t(\lambda), r(\lambda), \theta(\lambda), \phi(\lambda))$ . Spherical symmetry allows us to assume the orbit lies in the equatorial plane,  $\theta = \pi/2$ . Rotational symmetry  $\phi \rightarrow \phi + \Delta\phi$  and time translation symmetry  $t \rightarrow t + \Delta t$  give us conserved quantities which are the

momenta conjugate to  $\phi$  and  $t$  respectively,

$$p_\phi = mg_{\phi\mu}\dot{x}^\mu = mr^2\dot{\phi} \quad (21)$$

$$p_t = mg_{t\mu}\dot{x}^\mu = mg_{tt}\dot{t}. \quad (22)$$

When using the proper time parameter  $\lambda = \tau$ , these are (up to a sign in the case of the energy) the angular momentum and energy. Also when using the proper time parameter the 4-velocity is a unit vector,

$$-1 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + r^2\dot{\phi}^2 \quad (23)$$

where the diagonal nature of the Schwarzschild metric has been used. Using (21) and (22), and multiplying by  $g_{tt}/2$  and using  $g_{tt}g_{rr} = -1$  in Schwarzschild, we can express this in terms of an effective potential as

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r; \ell) = \frac{e^2 - 1}{2}, \quad (24)$$

$$V_{\text{eff}}(r; \ell) = -\frac{1}{r} + \frac{\ell^2}{2r^2} - \frac{\ell^2}{r^3}. \quad (25)$$

Here  $e = -p_t/m$  is the energy per unit mass,  $\ell = p_\phi/m$  is the angular momentum per unit mass, and I have used units with  $GM = c = 1$ . Other than an energy shift, the difference from the Newtonian case is the negative  $1/r^3$  term which always dominates at sufficiently small radius, creating a bottomless pit in the potential.

## Tuesday, Sept. 23

+ **Circular timelike orbits in Schwarzschild spacetime:** The proper time derivative of (24) yields  $\dot{r}\ddot{r} = -\dot{r}V'_{\text{eff}}(r)$ . For any nonzero  $\dot{r}$  we therefore have  $\ddot{r} = -V'_{\text{eff}}(r)$ . By continuity, the limiting case of a circular orbit should also satisfy this relation, which can also be checked directly from the geodesic equation. Thus at a circular orbit we must have  $V'_{\text{eff}}(r) = 0$ . This is equivalent to  $r^2 = \ell^2(r - 3)$ , or  $r_\pm(\ell) = \ell^2(1 \pm \sqrt{1 - 12/\ell^2})$ . For  $\ell > \ell_{\text{min}} = \sqrt{12}$  there is a stable circular orbit at  $r_+(\ell)$  as well as an unstable circular orbit at  $r_-(\ell)$  that lies closer to the horizon. As the angular momentum goes to infinity, the radius of the stable orbit goes to infinity and its energy goes to zero, while the radius of the unstable orbit shrinks to 3 and its energy goes to infinity. For angular momentum equal to  $\sqrt{12}$  the stable and unstable orbits merge at  $r_\pm = 6 = 6M$ . This is called the **innermost stable circular orbit**, or **ISCO**, but it is actually only marginally stable. For  $\ell < \sqrt{12}$  there are no circular orbits. This is unlike the Newtonian case in which arbitrarily

small circular orbits around a point mass exist.

+ **Lightlike (null) geodesics, affine parameter:** A lightlike curve has a null tangent vector, so  $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$  is zero, hence the proper time along such a curve is zero, and the variation of the proper time interval is ill-defined. Our derivation of the geodesic equation therefore doesn't apply to null curves. However, the equation (20) itself continues to make sense, albeit with a different interpretation of the parameter. Instead of the proper time  $\tau$ , we use a different letter, e.g.  $\lambda$ , and interpret the parameter as some other label of the points along the curve. For any parameter linearly related to  $\lambda$  the same equation will be satisfied. Any one of these parameters is called an **affine parameter**. A nonlinearly related parameter will introduce another term into the equation, proportional  $g_{\mu\alpha}\dot{x}^\mu$ . We will show later from Maxwell's equations that, in the ray optics approximation, electromagnetic waves follow lightlike (null) curves satisfying this geodesic equation. Similarly, photons can be considered massless particles that follow null geodesics.

+ One way to think of the affine parameter is via a limit from the timelike case: the 4-momentum of a massive particle is  $m dx^\mu/d\tau$ . In the limit  $m \rightarrow 0$ ,  $d\tau \rightarrow 0$ , the ratio  $m/d\tau$  is held fixed and is equal to  $1/d\lambda$  where  $\lambda$  is an affine parameter. A timelike 4-momentum has a zero-momentum frame, and the mass  $m$  is the energy in the that frame. So  $m$  sets the scale of the 4-momentum. For a lightlike 4-momentum there is no zero momentum frame, so the scale must be set some other way. The linear freedom in the affine parameter encodes this scale.

+ **Lightlike orbits:** For light like geodesics, tangent is a null vector so the left hand side of (23) is 0. The corresponding modification of (24) is that there is no  $-1/r$  term in  $V_{\text{eff}}$ , and the constant  $-1/2$  is missing from the right hand side. (Note that then the equation becomes invariant under a change of scaling of the affine parameter, as it must be.)

## Thursday, Sept. 25

Perihelion advance, light bending, Shapiro time delay.

## Tuesday, Sept. 30

Affine parameter, explanation of hw problem about horizon generator, redshift, Killing vectors (example of the translations and rotation on the Euclidean plane), energy and angular momentum in terms of Killing vectors, redshift of spectral lines

from orbiting material.

## Thursday, Oct. 2

+ Finished calculation of redshift/blueshift from orbiting atom. The limiting values for the ratio of frequency at infinity to proper frequency for a source atom on the ISCO orbit are  $\sqrt{2} = 1.41$  for approaching,  $1/\sqrt{2} = 0.71$  for transverse, and  $\sqrt{2}/3 = 0.47$  for receding cases. Explained shape of spectral line. For a spinning black hole, the ISCO is closer to the horizon, so the maximal redshift is greater. See <http://adsabs.harvard.edu/abs/2000PASP..112.1145F> for observational details. The paper is also linked at the supplements page of the course web page, <http://www.physics.umd.edu/grt/taj/675e/675esupplements.html>

This can be used to measure the spin of black holes. There are claims of spins very close to the maximal value having been measured. However Cole Miller of the UMD Astronomy Dept. cautions that because some emission can come from within the ISCO it is not at present possible to be certain that any spin is greater than about 80% of the maximal value.

+ Noted energy per unit mass of Schwarzschild ISCO is  $2\sqrt{2}/3 \approx 0.94$ . So the binding energy is 6% of the rest energy. This is a significant source of energy. For a spinning black hole, the ISCO is closer to the horizon, so the binding energy is greater.

+ Explained in what sense **a vector is equivalent to a directional derivative operator**, so that we sometimes express vectors as the corresponding partial derivative operator. For example, in Schwarzschild the Killing vectors are  $\partial_t$  and  $\partial_\phi$ . Note that the meaning of partial derivative depends on the coordinate system.  $\partial_t$  in Schwarzschild coordinates is the same as  $\partial_v$  in Eddington-Finkelstein coordinates. And  $\partial_r$  is different in the two coordinate systems.

+ Discussed aspects of gravitational lensing. Showed and discussed a presentation that can be found at the supplements page, <http://www.physics.umd.edu/grt/taj/675e/675esupplements.html>.

+ Started discussing the time reversed EF coordinate system, and the full Schwarzschild spacetime. To be completed next week.

## Tuesday, Oct. 7

A one solar mass Schwarzschild black hole has horizon radius of 2.95 km, so 3km is a good number to remember. In general then,

$$2GM/c^2 \approx 3\text{km} (M/M_\odot).$$

[*Did not mention this in class:* The ISCO radius at  $r = 6M$  is then about 9 km. A  $1.4M_\odot$  non-spinning neutron star would then have an ISCO radius of 12.6 km. It is not known whether this is greater than the radius of the star. However a  $2M_\odot$  neutron star would have an ISCO radius of 18 km. If the neutron star is spinning then the metric outside is not the Schwarzschild one, and the ISCO would be smaller. Still, there is evidence that at least some neutron stars are inside their ISCOs.]

+ To find a coordinate change that extends across the Schwarzschild horizon, consider radial light rays, which satisfy  $dt^2 = (1 - 1/r)^{-2} dr^2$ . [Here and below I use units with  $r_S = 2GM/c^2 = 1$ , and I suppress the angular coordinate part of the line element.] This is equivalent to  $d(t \pm r_*) = 0$ , where  $r_*$  is the **tortoise coordinate** defined by

$$dr_* = \frac{dr}{1 - 1/r}, \quad r_* = r + \ln(1 - r). \quad (26)$$

So introduce new coordinates,

$$v = t + r_* \quad \text{ingoing} \quad (27)$$

$$u = t - r_* \quad \text{outgoing} \quad (28)$$

A constant  $v$  line crosses the horizon, with  $t \rightarrow \infty$  and  $r_* \rightarrow -\infty$  at the horizon. Similarly for a constant  $u$  line, with  $t \rightarrow -\infty$  and  $r_* \rightarrow -\infty$ . This means the  $u$  line must be crossing “somewhere else”... The replacements  $dt = dv - dr_*$  or  $dt = du + dr_*$  in the Schwarzschild line element produce the *ingoing and outgoing Eddington-Finkelstein* line elements,

$$ds^2 = -(1 - 1/r)dv^2 + 2dv dr \quad \text{ingoing} \quad (29)$$

$$= -(1 - 1/r)du^2 - 2du dr \quad \text{outgoing}. \quad (30)$$

These both apply in the original Schwarzschild region, but the former extends across the future horizon and runs to finite affine parameter at  $v \rightarrow -\infty$ , and the latter extends across the past horizon and runs to finite affine parameter at  $u \rightarrow \infty$ . Neither covers the wedge outside both of these, as the diagram drawn in class made

clear. To cover everything at once, we need to find other coordinates.

+ Historical notes: Eddington in 1924 and, without knowing of Eddington's work, Finkelstein in 1958 introduced almost these coordinates, but instead of  $r_*$  used just  $\ln(r - 1)$ . Eddington did not remark that the  $r = 1$  surface was regular using this coordinate. Finkelstein did, and he called that surface a "uni-directional membrane" and drew a spacetime diagram. This was influential in helping physicists to understand the nature of the horizon. Kruskal's paper was written by Wheeler in 1960. Kruskal was a plasma physicist who had worked this out earlier in the '50's and never had published it. I need to find the reference to Szekeres' paper...

+ To arrive at **Kruskal-Szekeres coordinates**, we use both  $u$  and  $v$  as coordinates. Note that

$$r - 1 = e^{r_* - r} = e^{v/2} e^{-u/2} e^{-r}, \quad (31)$$

so we can write the line element as follows:

$$ds^2 = -(1 - 1/r) du dv \quad (32)$$

$$= -\frac{r-1}{r} dudv \quad (33)$$

$$= -\frac{4e^{-r}}{r} d(-e^{-u/2}) d(e^{v/2}) \quad (34)$$

$$= -\frac{4e^{-r}}{r} dU dV, \quad (35)$$

with

$$V = e^{v/2}, \quad U = -e^{-u/2}, \quad UV = -(r-1)e^r, \quad V/U = -e^t. \quad (36)$$

(N.B. This  $U$  and  $V$  is not the same as Hartle's.) (Note that if we put back in  $r_S$  we have for example  $V = \exp(v/2r_S) = \exp(\kappa v)$ , where  $\kappa = 1/2r_S = 1/4GM$  is the surface gravity of the black hole. Connecting with the homework problem, this shows that  $V$  is an affine parameter along the horizon null geodesics.) Now we can extend  $U$  forward past  $u = \infty$ , and extend  $V$  backward past  $v = -\infty$ . The quadrant that was missing before has  $U > 0$  and  $V < 0$ . This gives the maximal extension of the spacetime. One can check that all geodesics run for infinite affine parameter, unless they run into one of the singularities.

+ Constant  $r$  curves are hyperbolae in  $U, V$  space, while constant  $t$  lines are straight lines through the horizon. The time translation symmetry flows along constant  $r$  lines. I showed how it flows. It's the Lorentzian analog of a rotation. In

Minkowski spacetime, it would be the Lorentz boost symmetry. The corresponding Killing vector is

$$\partial_t = \partial_u + \partial_v = \frac{1}{2}(-U\partial_U + V\partial_V). \quad (37)$$

There is a future/black hole horizon, and a past/white hole horizon, and similarly for the  $r = 0$  singularities. There is a second asymptotically flat spatial region, connected to the original one through a “throat”. On a constant  $t$  slice, which goes through the point  $U = 0, V = 0$ , the throat has minimal spherical radius  $r_S$ .

+ **Penrose-Carter diagrams:** the infinite regions of the spacetime can be brought to a finite place by multiplying the metric by a conformal factor that goes to zero in the right way at infinity. I drew the diagram for the Schwarzschild spacetime, and introduced the concepts of future and past null infinity  $\mathcal{I}^\pm$ , called “scri-plus” and “scri-minus”, future and past timelike infinity  $i^\pm$ , and spacelike infinity  $i^0$ . Some handwritten lecture notes by Albert Roura on Penrose-Carter diagrams are posted at the supplements link of the course webpage.

+ **Collapse of a spherical mass shell to form a black hole:** inside the shell is Minkowski spacetime, outside is Schwarzschild. The horizon is born as a light cone in the center, that bends to the horizon when it crosses through the shell. I drew the spacetime diagram and explained it, and then drew the corresponding Carter-Penrose diagram. Instead of a shell we could also consider a collapsing ball of matter — a star. An observer outside would never see the shell fall across the horizon, because the outgoing radial light rays that leave the shell at that point don’t escape: they generate the horizon. The outside observer would see the process of fall to the horizon stretched out over an infinite time. The frequency of any radiation from the shell that reaches the outside observer would decrease to zero. In this sense the horizon is an **infinite redshift surface**.

## Thursday, Oct. 9

+ Let’s show that the rate of decay of radiation from an object that falls across a Schwarzschild black hole horizon is governed by the surface gravity of the black hole. To be specific, consider the collapsing shell described in the previous class. The trajectory of the shell is given by functions  $v(\tau), r(\tau)$ , where  $\tau$  is the proper time on the shell. Let’s set  $\tau = 0$  and  $v = 0$  at the horizon crossing point, and make a Taylor expansion around that point:  $v(\tau) = A\tau + O(\tau^2)$  and  $r(\tau) - 1 = -B\tau + O(\tau^2)$ , where  $A$  and  $B$  are positive constants. The outgoing light rays are lines of constant  $u$  (28), and at constant radius  $u$  measures Schwarzschild time  $t$ ,

so we want to find  $u(\tau)$  along the world line of the shell. Neglecting the  $O(\tau^2)$  terms we have

$$u(\tau) = v(\tau) - 2r_*(\tau) = A\tau - 2(1 - B\tau) - 2\ln(-B\tau). \quad (38)$$

As  $\tau \rightarrow 0^-$ , the log term diverges, and the other terms are finite, so  $u$  goes to infinity. To characterize this divergence we can set  $\tau = 0$  in the linear terms, after which solving for  $\tau$  yields  $\tau = -C'e^{-u/2}$  for some positive constant  $C'$ . The exponent should be dimensionless; since  $u$  has dimensions of length there is an implicit  $r_S$  in the denominator. The surface gravity is  $\kappa = 1/2r_S$ , so we have  $\tau = -C'e^{-\kappa u}$ , and hence  $\tau$  is related to the Schwarzschild time of an outgoing light ray at fixed  $r$  by

$$\tau = -Ce^{-\kappa t}, \quad (39)$$

where  $C$  is another positive constant. This exhibits logarithmic divergence of the outside observer's time as the shell crosses the horizon. If the observer sits at large  $r$  where Schwarzschild  $t$  agrees with the observer's proper time, the redshift is given by

$$\frac{\omega_{\text{obs}}}{\omega_{\text{em}}} = \frac{d\tau}{dt} = \kappa C e^{-\kappa t}. \quad (40)$$

This is the exponential redshift. The shell “disappears” to the outside observer in this exponential fashion. For this reason, Brandon Carter called  $\kappa$  the **decay constant** of the black hole.

**+ Field of a body with angular momentum:** For weak fields we can use linearity and Lorentz invariance to infer how the spin of a source affects the metric. First consider the electromagnetic analogy. Consider a source with current density  $\mathbf{j} = \rho\mathbf{v}$ . To first order in the velocity of the charge we can view this at each point as the result of a charge density  $\rho$  viewed in a frame moving with velocity  $-\mathbf{v}$  relative to the source. The vector potential it contributes can thus be obtained by Lorentz transformation of the purely scalar Coulomb potential  $\rho/|\mathbf{x} - \mathbf{x}'|d^3x'$ . In terms of the 4-vector potential we have  $A_i = (\partial x'^\alpha/\partial x_i)A'_\alpha = (\partial t/\partial x^i)A_t = \gamma v_i A_t$ . *I am not trying to keep track of the signs here.* At linear order in  $v$  this gives  $\mathbf{A} = \mathbf{v}\phi$ , where  $\phi$  is the Coulomb potential, so for our source,  $d\mathbf{A} = \rho\mathbf{v}/|\mathbf{x} - \mathbf{x}'|d^3x'$ . Integrating over the source yields the vector potential  $\mathbf{A} = \int \mathbf{j}/|\mathbf{x} - \mathbf{x}'|d^3x'$ , which is indeed what Maxwell's equations imply in a suitable gauge (up to the sign).

**Warning:** The following may be wrong. The result is off by a factor of 2 which may indicate an incorrect method for reason's I've not yet understood. For gravity it's similar, but  $\rho$  becomes the mass density, and we need to take the Lorentz transformation of the metric perturbation. The weak field metric of a spherical source can be written as  $g_{\mu\nu} = \eta_{\mu\nu} + (2m/r)(\delta_\mu^t \delta_\nu^t + \delta_\mu^r \delta_\nu^r)$ . If we assume, as

seems natural, that the  $m$ -dependent part can be superposed when we have a weak source, it seems we should be able to make a Lorentz transformation on this part and add up the pieces to get the correction due to motion of the source, e.g. angular momentum.

Let's consider the case of a ring of radius  $R$  and mass  $M$  rotating in the  $xy$  plane with tangential speed  $v$ , and let's take the field point far away on the  $x$  axis, so only the component of velocity in the  $y$  direction will contribute to the perturbation after integrating over the ring. We use a Minkowski coordinate system. The Lorentz transform from the frame of a moving mass element introduces a  $ty$  component to the metric,  $g_{ty} = (\partial x'^\alpha / \partial t)(\partial x'^\beta / \partial y)(2m/r)(\delta_\alpha^t \delta_\beta^t + \delta_\alpha^x \delta_\beta^x) = (\partial t' / \partial t)(\partial t' / \partial y)(2m/r) = 2mv_y/r$ . Now say  $r$  is the distance from field point to the center of the ring. Integrate over the ring from  $\phi = -\pi/2$  to  $\pi/2$ . Opposite sides of the ring contribute with opposite  $v_y$ , and the distances to the field point are, to leading order in  $1/r$ ,  $1/(r - R \cos \phi)$  and  $1/(r + R \cos \phi)$ , whose difference is to leading order  $2R \cos \phi / r^2$ . So the integral we must do to compute  $g_{ty}$  is  $\int_{-\pi/2}^{\pi/2} 4M(d\phi/2\pi)(R/r^2)(v \cos \phi) \cos \phi = MvR/r^2 = J/r^2$ , where  $J = MvR$  is the angular momentum of the ring. Since  $dy = r d\phi$ , we have  $g_{t\phi} = r g_{ty} = J/r$ , and so the contribution to  $ds^2$  is  $(2J/r) dt d\phi$ . This is too small by a factor of 2. So far I don't see any conceptual error here...but I suspect there is one, since I can't find a computational error...

## Tuesday, Oct. 14

+ All about the **Kerr metric**. Much of what we covered is covered in Hartle's book pretty much the same way. One difference is that I wrote the metric in two ways, one of which Hartle wrote, the other being

$$ds^2 = -\frac{\rho^2 \Delta}{A} dt^2 + \frac{A \sin^2 \theta}{\rho^2} (d\phi - \Omega_Z dt)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \quad (41)$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2 \quad (42)$$

$$A = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad \Omega_Z = \frac{2Mar}{A} \quad (43)$$

+ A null surface is one that is everywhere tangent to the light cone. Put differently, it contains one null direction at each point, and the rest of the directions are spacelike. Through energy point on a null surface there is thus a null "generating curve", the **null generator**. A vector tangent to the generator is orthogonal to itself and to every other vector tangent to the surface. So the normal vector is null. This

is another way to identify a null surface. If the surface is defined by a level set of a function,  $f(x) = C = \text{constant}$ , then the gradient  $\partial_\mu f$  is null in the sense that  $g^{\mu\nu} \partial_\mu f \partial_\nu f = 0$ , where  $g^{\mu\nu}$  is the inverse metric. This is equivalent to saying that the vector  $g^{\mu\nu} \partial_\mu f$  is a null vector.

We are interested in finding null surfaces in the Kerr spacetime which are invariant under the Killing symmetries. Such surfaces should be defined by a level set of a function  $f(r, \theta)$  that is independent of  $t$  and  $\phi$ . In Boyer-Lindquist coordinates we have  $g^{rr} = \Delta/\rho^2$  and  $g^{\theta\theta} = 1/\rho^2$ . The condition for our invariant null surface is thus  $\Delta f_{,r} f_{,r} + f_{,\theta} f_{,\theta} = 0$ . This is satisfied for a constant  $r$  surface where  $\Delta = 0$ , i.e.  $r = r_\pm = M \pm \sqrt{M^2 - a^2}$ .<sup>2</sup> These are the inner and outer horizons.<sup>3</sup> It is now easy to identify the angular velocity of the black hole  $\Omega_H$ , which is defined as the angular velocity of the null generators of the horizon. Looking at the line element (41), you see that when  $\Delta = 0$  and  $dr = 0$ , the only way to have  $ds^2 = 0$  is to have  $d\phi = \Omega_Z dt$  and  $d\theta = 0$ . On the horizon we have  $\Omega_Z = \Omega_H = a/2Mr_\pm = a/(r_\pm^2 + a^2)$ .

**+ Area of the horizon:** The horizon is a null surface, and the null direction is a symmetry direction, so the 2d geometry of all cross-sections is identical. A cross-section is coordinated by  $(\theta, \phi)$ . The area of a cross section of the outer horizon is  $\int d\theta d\phi \sqrt{g_{\theta\theta} g_{\phi\phi}} = \int d\theta d\phi \sqrt{A} \sin \theta = 4\pi(r_+^2 + a^2) = 8\pi Mr_+$ .

## Thursday, Oct. 16

+ Orbits of Kerr: showed a simulation; the Mathematica notebook file is linked at the supplements page. Discussed the ISCO: it moves closer to the horizon as the spin parameter  $a = J/M$  increases. At the extremal limit  $a = M$ , the ISCO coincides with the horizon when examined for example by an in falling observer. On a surface of constant Boyer-Lindquist time, the distance to the horizon diverges in the extremal limit, as does the distance to the ISCO, but the ISCO remains at a finite distance from the horizon.

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<sup>2</sup>It can also be satisfied on a surface where  $\Delta < 0$  and  $f_{,\theta} = \pm f_{,r} \sqrt{-\Delta}$ . For example choose  $f(r, \theta) = \theta(r)$ , with  $d\theta/dr = (-\Delta)^{-1/2}$ . In Schwarzschild spacetime, where  $-\Delta = r(2M - r) > 0$  inside the horizon, a solution is  $\theta = 2 \sin^{-1}(\sqrt{r/2M})$ . This ranges from 0 to  $\pi$  as  $r$  ranges from 0 to  $2M$ .

<sup>3</sup>Actually we should check that the vector  $f_{,\mu}$  is a *non-vanishing* vector — otherwise it is null but in a trivial way that does not identify a null surface. For the case  $f(r) = r$ ,  $r_{,\mu}$  is clearly *not* the zero vector, because for example an infalling timelike geodesic can have  $\dot{r} = u^\alpha r_{,\alpha} < 0$  at any value of  $r$  including  $r_\pm$ . For an example where it is zero, consider just the spatial section of the Schwarzschild metric: then  $\Delta = 0$  at the minimal area sphere, and the gradient of  $r$  vanishes at that sphere.

+ There is a miraculous constant of the motion, the **Carter constant**, that makes the orbital equation integrable, despite the presence of only two symmetries (time translation and rotation about the symmetry axis). A related miracle is that the wave equations for scalars, spin-1/2, spin-1, and spin-2 fields all separate in known coordinate systems for Kerr. In fact, Carter found his constant of the motion as a separation constant for separable solutions of the Hamilton-Jacobi equation, which is I suppose the WKB approximation of these wave equations.

+ **Penrose process**: Negative Killing energy states exist in the ergosphere, where the Killing vector  $\partial_t$  is spacelike outside the horizon. This can be exploited to extract rotational energy from a spinning black hole. The original article where Penrose proposed this is linked at the course supplements page. It has a number of terrific diagrams in it. He described lowering a particle into a negative energy orbit from a rotating scaffolding. The scaffolding must be rotating because the particle must be rotating when it is in the ergosphere, inside the stationary limit surface. He also mentioned a ballistic method where a particle breaks up into two in the ergosphere, one with positive and one with negative energy. A black hole with a surrounding plasma threaded by a magnetic field has a sort of continuous Penrose process, the Blandford-Znajek mechanism. This might be the source of jet power from active galactic nuclei.

+ **Efficiency** of the Penrose process: if a particle crosses the horizon with 4-momentum  $p$ , then  $p \cdot (\partial_t + \Omega_H \partial_\phi) \leq 0$ , because  $p$  is future timeline and the Killing vector is future null. The Killing energy and angular momentum of the particle thus satisfy  $\delta E - \Omega_H \delta L \geq 0$ . Note  $\delta E$  and  $\delta L$  must both be negative, and the negative energy is maximized negatively for a given angular momentum change when the inequality is saturated, i.e. when  $p$  is tangent to the horizon null generator. Thus in any such process, the energy and angular momentum change of the black hole must satisfy  $\delta M - \Omega_H \delta J \geq 0$ , and the most efficient extraction is when this is zero.

+ **“First law of black hole mechanics”** for Kerr black holes:  $\delta M - \Omega_H \delta J = \frac{\kappa}{8\pi G} \delta A$ . The fact that the area change must be positive is more general than this, and is called the “area theorem”. Note that the most efficient extraction occurs in the reversible case when the area is unchanged.

+ Here’s a proof of the first law for Kerr black holes, using the relations be-

tween the parameters of the Kerr metric:

$$\delta M - \Omega_H \delta J = \frac{1}{2Mr_+} (r_+^2 \delta M - Ma \delta a) \quad (44)$$

$$= \frac{r_+ - M}{2Mr_+} (r_+ \delta M + M \delta r_+) \quad (45)$$

$$= \frac{r_+ - M}{2Mr_+} \delta(Mr_+) \quad (46)$$

$$= \frac{\kappa}{8\pi G} \delta A. \quad (47)$$

Uses in the first line  $J = aM$ ,  $\Omega_H = a/2Mr_+$ , and  $r_+^2 = 2Mr_+ - a^2$ ; in the second line  $\delta(a^2 = 2Mr_+ - r_+^2)$ ; in the last line  $\kappa = (r_+ - M)/(2Mr_+)$  and  $A = 8\pi Mr_+$ .

+ **First Law of black hole mechanics more generally:** Bardeen, Carter and Hawking proved the first law in greater generality using the Einstein field equation and general properties of the horizon, including the intrinsic definition of surface gravity. Even more generally, Sudarsky and Wald showed that one can trace the origin of this identity to the general covariance of the theory, which implies that when the field equations hold the Hamiltonian is a boundary term. The left hand side of the first law is a boundary term at infinity, while the area variation term is a boundary term at the horizon. One can show that when making a variation away from a stationary solution, the sum of the two boundary term variations vanishes. The implied relation between the variations of the boundary terms at the horizon and at infinity is the first law. Other generally covariant gravity theories have similar relations, with a different value for the “entropy”.

## Tuesday, Oct. 21

+ **Contravariant vectors** are four-component objects that transform like the prototype  $dx^\mu$ , i.e. with a factor of the Jacobian  $J$ . **Covariant vectors** transform like the prototype  $\partial f/\partial x^\mu$ , with a factor of the *inverse* Jacobian  $J^{-1}$ . We use superscripts for contravariant, and subscripts for covariant indices. The contraction  $\partial_\mu f dx^\mu$  is thus invariant, because  $J$  and  $J^{-1}$  combine to give the identity. This invariance is expected, since the contraction is equal to  $df$ , which is manifestly independent of coordinates. It is general: the **contraction** of a contravariant index with a covariant index is a coordinate invariant operation. **Tangent vectors** to a curve  $dx^\mu/d\lambda$  are vectors, and at each point these comprise what is called the **tangent space**. Covectors comprise the **cotangent space**.

+ Higher **rank** tensors are obtained from sums of outer products, and transform with one  $J$  for each contravariant index and one  $J^{-1}$  for each covariant index. Tensors of the same rank form a vector space at each point. Contraction of a co-contravariant index pair on a tensor gives a new tensor.

+ The metric is a tensor. To show this, note that  $ds^2$  is a scalar, so let's transform and see what that implies about the metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g'_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} dx^\alpha dx^\beta. \quad (48)$$

Since this is true for all  $dx^\mu$ , and  $g_{\mu\nu}$  is symmetric, this implies that  $g_{\alpha\beta} = g'_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta}$ , which is the covariant tensor transformation law (here  $x'$  plays the role of the original coordinates and  $x$  the new coordinates). The inverse metric and the Kronecker delta are also tensors. Contraction with the metric converts a contravariant index into a covariant one. This is called **lowering the index**. Similarly contraction of a covariant index with the inverse metric is **raising the index**.

+ **Quotient rule for tensoriality**: If  $V^\mu \omega_\mu$  is a scalar for all vectors  $V^\mu$ , then  $\omega_\mu$  is a covector. (Show this). This can be generalized to higher rank tensors.

+ **Covariant derivative**: The partial derivative of a vector is not a vector, because when it is transformed the derivative hits the Jacobian. Geometrically, the problem is that you can't meaningfully subtract vectors at two different points. They live in different vector spaces. If you try to just subtract them by subtracting their components in a given coordinate system, the result will not transform as a vector when you change coordinate systems. With a metric, we have a special class of coordinate systems associated to each point, namely the local inertial coordinates at that point. We can use that to define a **covariant derivative** as follows.

Any two local inertial coordinate systems at  $p$  are related by a Jacobian that has vanishing partial derivative at  $p$ , so the partial derivative of the vector components transforms as a tensor at  $p$  when changing from one local inertial frame to another. When changing to a generic coordinate system, something else results. I claim that in general it is given by:

$$\nabla_\mu T_\beta^\alpha = \partial_\mu T_\beta^\alpha + \Gamma^\alpha_{\mu\sigma} T_\beta^\sigma - \Gamma^\tau_{\mu\beta} T_\tau^\alpha, \quad (49)$$

where  $\Gamma^\alpha_{\mu\nu}$  is the **Christoffel symbol**.

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}). \quad (50)$$

More generally, there is one  $\Gamma$  term for each contravariant index and one  $-\Gamma$  term for each covariant index. This is called the (metric) **covariant derivative**. In a

local inertial coordinate system at a point  $p$ ,  $\Gamma(p) = 0$ , so (49) reduces to the partial derivative. It remains to show that (49) defines a tensor. We'll do this below. Note that  $\nabla_\mu g_{\alpha\beta}$  vanishes in a local inertial coordinate system, so it vanishes as a tensor. Thus the metric is “covariantly constant” with respect to this covariant derivative.

## Thursday, Oct. 23

+ The **inverse metric** is a tensor. I don't think the way I showed this in class was logically complete. Instead, one can just check that if it is transformed as a contravariant tensor then it will indeed be the inverse in the new coordinate system. Then since the inverse is unique, this shows that it is a tensor. Viz:

$$(g^{ab} x_{,a}^{c'} x_{,b}^{d'}) g'_{dn} = (g^{ab} x_{,a}^{c'} x_{,b}^{d'}) (x_{,d'}^r x_{,n'}^s g_{rs}) = \delta_n^c. \quad (51)$$

Here I use the slick notation  $x_{,a}^{c'} = \partial x^{c'}/\partial x^a$ . The central Jacobian and inverse Jacobian to combine to  $\delta_b^r$ , which then contracts with the metric and inverse metric to give  $\delta_s^a$ , which then contracts with the remaining Jacobian and inverse Jacobian to give the result.

+ The presence of the Christoffel symbol (50) in the covariant derivative (49) can be motivated by its role in the geodesic equation,

$$a^\mu \equiv \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0. \quad (52)$$

Neither term in this expression is a vector, but the sum is a vector, called the (co-)variant **acceleration**. Without cranking out the coordinate transformation, we can infer that it is a vector as follows. The geodesic equation is equivalent to the condition that the integral  $\int \sqrt{-ds^2}$  be stationary, together with the affine parameter condition. Those conditions are independent of coordinates, so the geodesic equation must be as well, which strongly suggests that the acceleration is a vector. In fact it *is* a vector. One way to argue that is to note that the variation of the action  $S = \int \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta d\lambda$  is given by  $\delta S = \int E_\mu \delta x^\mu d\lambda$ , with  $E_\mu = -g_{\mu\nu} a^\nu$ . Since  $\delta S$  is a scalar for all variations  $\delta x^\mu$ , and  $\delta x^\mu$  is a vector (*do you know why?*), the quotient rule implies that  $E_\mu$  is a vector, which implies that  $a^\mu$  is a vector, since it is obtained from a vector by contraction with the inverse metric (which is a tensor). It is possible to leverage the vector character of the acceleration to argue that the covariant derivative of a tensor is a tensor. But rather than going that route, I want to show it using a more general argument which will at the same time explain a more universal concept of covariant derivative.

+ Suppose I have a field  $\psi(x)$  that takes values in some vector space  $V_x$  at each  $x$ . Of course an example is just a vector field on spacetime, the vector space  $V_x$  being the tangent space at  $x$ . But  $V_x$  could also be a vector space that has nothing to do with spacetime, for instance the complex three dimensional space of quark colors, in which case  $\psi$  would be a quark field. Under a coordinate change, a tangent vector transforms by the Jacobian  $J$ , viz.  $V \rightarrow J(x)V$ . Similarly, the theory of quarks, QCD, has a symmetry under local “rotations” in color space, or more precisely local  $SU(3)$  transformations. These have the effect on a quark field

$$\psi \rightarrow g(x)\psi, \quad (53)$$

where  $g(x)$  is an  $SU(3)$  group element which can be represented by a  $3 \times 3$  matrix that multiplies the 3-component complex vector  $\psi$ . Now let’s see how a derivative of  $\psi$  transforms:

$$\partial_\mu \psi \rightarrow \partial_\mu (g\psi) = g\partial_\mu \psi + (\partial_\mu g)\psi. \quad (54)$$

The derivative does not transform the same way as  $\psi$  does. If we are to construct a theory that is invariant under local  $SU(3)$  transformations, we should work with quantities that transform “covariantly” as does  $\psi$ . This can be achieved by replacing the derivative operator  $\partial_\mu$  by a **gauge-covariant derivative**  $D_\mu$ , which acts on  $\psi$  as

$$D_\mu \psi = (\partial_\mu + B_\mu)\psi, \quad (55)$$

where the **connection**  $B_\mu$  is a  $3 \times 3$  matrix whose transformation property is to be determined. Let’s see how the action of the new derivative operator transforms:

$$(\partial_\mu + B'_\mu)g\psi = g(\partial_\mu + B_\mu)\psi + (B'_\mu g - gB_\mu + \partial_\mu g)\psi. \quad (56)$$

The transformation will be homogeneous if we define  $B'_\mu$  so as to kill off the second term,

$$B'_\mu = gB_\mu g^{-1} - (\partial_\mu g)g^{-1}. \quad (57)$$

This defines the transformation law for a general connection.

We used nothing special about the group  $SU(3)$ , so it could be any other Lie group. For example, it could be  $U(1)$ , the group of complex phases  $e^{iq\theta}$  for a charge  $q$ , in which case (56) becomes  $B'_\mu = B_\mu - iq\partial_\mu\theta$ . If we define  $A_\mu$  by  $B_\mu = iqA_\mu$ , this becomes  $A'_\mu = A_\mu - \partial_\mu\theta$ , the gauge transformation of an electromagnetic “vector potential”, and the derivative operator becomes  $\partial_\mu + iqA_\mu$ . The use of this derivative operator enables the theory to have local phase invariance.

Now coming back to the Christoffel symbol, this too is a connection, and the local group is the linear transformations in the tangent space. To make contact with the above general discussion, we can write

$$(B_\mu)_\beta^\alpha = \Gamma^\alpha_{\mu\beta}. \quad (58)$$

So the covariant derivative of a tensor will be a tensor provided the  $\alpha_\beta$  indices on the Christoffel symbol  $\Gamma^\alpha_{\mu\beta}$  transform as in (56) under a coordinate transformation, with  $g$  replaced by the Jacobian  $J$ . Checking this transformation behavior is one of those things where good notation and good organization really helps to avoid an explosion of terms and writing. I invented a notation to save some writing: Let

$$x_{,\tau}^{\alpha'} = \frac{\partial x'^{\alpha}}{\partial x^\tau}. \quad (59)$$

Note that I drop the  $\partial$ 's, write only one  $x$ , and attach the prime to the index rather than to the  $x$ . I also use this index rule on other quantities: an index denoting components in a primed coordinate system carries a prime. I leave the details to a footnote.<sup>4</sup>

+ **Curvature:** The quick way:

$$(D_\mu D_\nu - D_\nu D_\mu)\psi = \Omega_{\mu\nu}\psi, \quad (64)$$

where

$$\Omega_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + B_\mu B_\nu - B_\nu B_\mu. \quad (65)$$

To see this, check that all terms where one or two partial derivatives hit  $\psi$  vanish upon antisymmetrization of  $\mu$  and  $\nu$ . Note that from the known transformation behavior of the left hand side of (64) we infer that  $\Omega_{\mu\nu} \rightarrow g \Omega_{\mu\nu} g^{-1}$ .

+ For the electromagnetic connection  $\Omega_{\mu\nu} = iq(\partial_\mu A_\nu - \partial_\nu A_\mu) = iqF_{\mu\nu}$ , where  $F_{\mu\nu}$  is the usual electromagnetic **field strength tensor**. So the field strength is the curvature of a  $U(1)$  connection. It is gauge invariant because the  $U(1)$  group is abelian (commutative).

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<sup>4</sup> OK then, here goes:

$$\Gamma^{\alpha'}_{\mu'\beta'} = \frac{1}{2}g^{\alpha'\gamma'}[(g_{\sigma\tau}x_{,\gamma'}^\sigma x_{,\mu'}^\tau)_{,\beta'} + (\mu' \leftrightarrow \beta') - (g_{\sigma\tau}x_{,\mu'}^\sigma x_{,\beta'}^\tau)_{,\gamma'}] \quad (60)$$

$$= x_{,\rho}^{\alpha'} x_{,\beta'}^\sigma x_{,\mu'}^\tau \Gamma^\rho_{\sigma\tau} + g^{\alpha'\gamma'} g_{\sigma\tau} x_{,\gamma'}^\sigma x_{,\mu'}^\tau x_{,\beta'}^\rho \quad (61)$$

$$= x_{,\rho}^{\alpha'} x_{,\beta'}^\sigma x_{,\mu'}^\tau \Gamma^\rho_{\sigma\tau} + x_{,\tau}^{\alpha'} x_{,\mu'}^\tau x_{,\beta'}^\rho \quad (62)$$

$$= x_{,\mu'}^\tau [JB_\tau J^{-1} - (\partial_\tau J)J^{-1}]_{\beta'}^{\alpha'}. \quad (63)$$

Let me explain what happened here. In the second line, the terms where there are no derivatives of the Jacobian are gathered together into just what we would have if  $\Gamma^{\alpha'}_{\mu'\beta'}$  were a tensor. If you stare at the rest of the terms a bit, you notice that of the six potential terms, two are identical and give what is written, while the other four cancel because  $g_{\sigma\tau}$  is symmetric and so the  $\sigma$  and  $\tau$  on the  $x$ 's can be swapped. In the third line I used the fact that if you contract  $g^{\alpha'\gamma'} g_{\sigma\tau} x_{,\gamma'}^\sigma$  with  $x_{,\delta'}^\tau$  you'd have  $g^{\alpha'\gamma'} g_{\gamma'\delta'} = \delta_{\delta'}^{\alpha'}$ , so the former is the inverse of the latter. Finally in the last line I just renamed the Jacobian by the letter  $J$  and used a matrix notation, and also integrated by parts on the last term and commuted the two partial derivatives.

## Tuesday, Oct. 28

+ Mentioned the footnote on the coordinate transformation of the Christoffel symbol.

+ Covariant derivative of other tensors: in general the action of  $D_\mu$  depends on what it acts on, just as is the case for the spacetime covariant derivative. On “vectors” it is defined by (55), the vector index being implicit in the notation.

+ Curvature: the commutator  $[D_\mu, D_\nu]$  depends on what it acts on. For example, on a scalar function, the covariant derivatives commute:  $\nabla_\mu \nabla_\nu f = \nabla_\mu \partial_\nu f = \partial_\mu \partial_\nu f - \Gamma^\sigma_{\mu\nu} f$ . The mixed partials commute, and the Christoffel symbol is symmetric in the lower indices, so this is the same as  $\nabla_\nu \nabla_\mu f$ . For the covariant derivative of a tensor, there is a **Riemann curvature tensor** contraction for each tensor index, e.g.

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) T_\beta^\alpha = R^\alpha_{\sigma\mu\nu} T_\beta^\sigma - R^\sigma_{\beta\mu\nu} T_\sigma^\alpha. \quad (66)$$

The Riemann tensor is an example of the curvature (65) for a general connection. Explicitly,

$$R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\nu\beta} + \Gamma^\alpha_{\mu\sigma} \Gamma^\sigma_{\nu\beta} - (\mu \leftrightarrow \nu) \quad (67)$$

We’ll come back to discuss the properties of the Riemann tensor after first discussing the vacuum field equations.

+ **Parallel transport:** Given a curve, with tangent vector  $V^\mu = \partial x^\mu / \partial \lambda$ , we can define a rule for “parallel transporting” a field  $\psi$  along the curve:

$$V^\mu D_\mu \psi = 0 = \frac{\partial}{\partial \lambda} \psi + V^\mu B_\mu \psi. \quad (68)$$

This is a first order ODE; given an initial  $\psi$  at a point on the curve it determines a unique  $\psi$  at all other points. In the case of the Levi-Civita connection (i.e. the one defined with the Christoffel symbol) we are transporting vectors or tensors around.

+ **Curvature and parallel transport around a closed loop** Curvature measures the failure of transport to be integrable, meaning that the transport from point  $p$  to point  $q$  depends on the path joining  $p$  to  $q$ . Put differently, transport around a closed loop does not return the field to its initial value. You can easily visualize this property for the case of parallel transport of tangent vectors on the sphere using the Levi-Civita connection: this transport preserves the inner products of vectors, and therefore preserves the norms. It also preserves the inner product with the tangent to a geodesic, hence it preserves the angle. With this information you can see how

to parallel transport a vector along a great circle on the sphere. Making a triangle out of three great circles you can see that transport along a closed loop results in a rotation. The angle of rotation depends on the curvature and the area enclosed.

There is a simple relation between the curvature and the transport around an infinitesimal coordinate rectangle. Suppose the rectangle lies at fixed values of all coordinates except two,  $s$  and  $t$ , and runs along the  $s$  and  $t$  coordinate lines in the range  $(0, s)$  and  $(0, t)$ . Then the net change of the field  $\psi$  is

$$\Delta\psi = -st\Omega_{st}\psi. \quad (69)$$

I'll sketch the demonstration of this. Consider the transport defined by ((68)) along the first edge of the coordinate rectangle from  $(0, 0)$  to  $(s, 0)$ . If  $B_s$  were constant the exact solution to the transport equation would be  $\exp(-sB_s)\psi = [1 - sB_s + s^2B_s^2/2 + \dots]\psi$ . Taking into account the  $s$ -dependence of  $B_s$ , the solution up to  $O(s^2)$  is

$$[1 - sB_s(s/2, 0) + s^2B_s^2/2]\psi, \quad (70)$$

where in the last term  $B_s$  can be evaluated at  $s = 0$ . (You should check this.) Thus the transport around the rectangle is given by

$$[1 + tB_t(0, t/2) + t^2B_t^2/2][1 + sB_s(s/2, t) + s^2B_s^2/2] \quad (71)$$

$$\times [1 - tB_t(s, t/2) + t^2B_t^2/2][1 - sB_s(s/2, 0) + s^2B_s^2/2]\psi. \quad (72)$$

The  $O(s)$  and  $O(t)$  terms cancel, as do the  $O(s^2)$  and  $O(t^2)$  terms. We get  $O(st)$  terms both from the algebraic cross-terms and from Taylor expansion of  $B_s(s/2, t)$  and  $B_t(s, t/2)$ . The result is

$$[1 + st(B_{s,t} - B_{t,s} + [B_t, B_s])]\psi = [1 + st\Omega_{ts}]\psi, \quad (73)$$

which establishes (69).

**+ Newtonian tidal tensor and vacuum field equation** In Newtonian gravity a freely falling particle has acceleration equal to the gravitational acceleration vector  $g^i = -\partial^i\Phi$  where  $\Phi$  is the gravitational potential. (The index on *partial*<sup>*i*</sup> is raised with the inverse Euclidean spatial metric.) The relative acceleration of two particles separated by a small vector  $S^j$  is thus  $S^j\partial_jg^i + O(S^2)$ . This is the same as the second time derivative of the separation vector, hence we have

$$\frac{d^2S^i}{dt^2} = v^i{}_jS^j + O(S^2), \quad v^i{}_j \equiv \partial_jg^i. \quad (74)$$

The tensor  $v^i{}_j$  (Greek letter lower case ‘‘upsilon’’) is the **Newtonian tidal tensor**. Note that it is *symmetric*,  $v_{ij} = v_{ji}$ , since the mixed partials of the potential commute. The Newtonian potential satisfies  $\nabla^2\Phi = 0$  in vacuum, which is equivalent

to the statement that the tidal tensor is traceless,

$$v^i{}_i = 0. \quad (75)$$

This is the Newtonian vacuum field equation. It has the following nice geometrical interpretation. The separation vector for test particles that are initially at rest with respect to each other at time  $t = 0$  is given by integrating (74),

$$S^i(t) = (\delta^i{}_j + \frac{1}{2}t^2 v^i{}_j) S^j(0) + O(S^2, t^3). \quad (76)$$

The determinant of this linear transformation of  $S^j$  is  $1 + \frac{1}{2}t^2 v^i{}_i + O(t^4)$ , so (75) implies that at  $O(t^2)$  the determinant is unity. The determinant of a linear transformation is the factor by which it transforms volumes, so the Newtonian tidal deviation is volume preserving at  $O(t^2)$ . For example, a sphere of test particles will deform to an ellipsoid of the same volume.

**+ Einsteinian tidal tensor and vacuum field equation:** To define general relativistic tidal tensor we consider a one-parameter family of geodesics  $x^\mu(s, t)$ . For each fixed  $s$  this is a geodesic with affine parameter  $t$ , while for each  $t$  it is a curve connecting the geodesics at a common proper time. The tangent vector to the geodesics is  $T^\mu = x^\mu{}_{,t}$ , and the tangent vector to the connecting curve is  $S^\mu = x^\mu{}_{,s}$ . This vector plays the role of the separation vector between two freely falling particles in the Newtonian treatment above. The second covariant derivative  $(T^m \nabla_\mu)(T^n \nabla_\nu)S^\sigma$  is the relative acceleration of two infinitesimally neighboring geodesics.

To compute this we need to move all derivatives onto  $T$  and use the geodesic equation  $(T^m \nabla_\mu T^\sigma = 0)$ . To this end, note that in a local inertial coordinate system we have  $T^\nu \nabla_\nu S^\sigma = x^\sigma{}_{,st} = x^\sigma{}_{,ts} = S^n \nabla_\nu T^\sigma$ . Since these are tensors they must therefore be equal in all coordinate systems<sup>5</sup>, so we have

$$T^\nu \nabla_\nu S^\sigma = x^\sigma{}_{,st} = x^\sigma{}_{,ts} = S^\nu \nabla_\nu T^\sigma. \quad (77)$$

Using this relation, we now compute the relative acceleration:

$$(T^\mu \nabla_\mu)(T^\nu \nabla_\nu)S^\sigma = (T^\mu \nabla_\mu)(S^\nu \nabla_\nu T^\sigma) \quad (78)$$

$$= (T^\mu \nabla_\mu S^\nu) \nabla_\nu T^\sigma + T^\mu S^\nu \nabla_\mu \nabla_\nu T^\sigma \quad (79)$$

$$= (S^\mu \nabla_\mu T^\nu) \nabla_\nu T^\sigma + T^\mu S^\nu \nabla_\mu \nabla_\nu T^\sigma \quad (80)$$

$$= (-S^\mu T^\nu + T^\mu S^\nu) \nabla_\mu \nabla_\nu T^\sigma \quad (81)$$

$$= T^\mu S^\nu (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) T^\sigma \quad (82)$$

$$= R^\sigma{}_{\tau\mu\nu} T^\tau T^\mu S^\nu. \quad (83)$$

<sup>5</sup>In a general coordinate system there's a  $\Gamma^\sigma{}_{\nu\lambda} T^\nu S^\lambda$  term which is also symmetric in  $T$  and  $S$ .

(I used the product rule in the second line, and integration by parts together with the geodesic equation in the fourth line.) The equality of the first and last members of this string of equations is called the **geodesic deviation equation**. It characterizes the curvature of the geometry by the relative acceleration of neighboring geodesics.

The geodesic deviation equation identifies the Einstein tidal tensor as

$$\Upsilon^\sigma{}_\nu = R^\sigma{}_{\tau\mu\nu} T^\tau T^\mu. \quad (84)$$

Unlike in Newtonian gravity, the tidal tensor depends on the 4-velocity  $T^\mu$  of the geodesic. While  $\Upsilon^\sigma{}_\nu$  is a spacetime tensor, it satisfies  $\Upsilon^\sigma{}_\nu T^\nu = 0 = \Upsilon^\sigma{}_\nu T_\sigma$ , since the Riemann tensor is antisymmetric on the second index pair as well as on the first index pair when the upper index is lowered (we didn't show this yet but we will). Thus it is zero except in the spatial subspace orthogonal to  $T^\mu$ .

The trace of the tidal tensor is given by

$$\Upsilon^\sigma{}_\sigma = R_{\mu\nu} T^\mu T^\nu, \quad (85)$$

where

$$R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu} \quad (86)$$

is the **Ricci tensor**. Now here comes the moment of truth: suppose we demand that Newton's vacuum field equation—tracelessness of the tidal tensor—holds *for all geodesics*. Then it follows that the Ricci tensor must vanish,

$$R_{\mu\nu} = 0. \quad (87)$$

This is **Einstein's vacuum field equation**. It is remarkable that it is even possible to impose tracelessness of the tidal tensor for all geodesics, and that this "simple" generalization of Newton's field equation gives Einstein's.

## Thursday, Oct. 30

+ **Nature of the vacuum field equation:** The vacuum equation (87) comprises 10 second order, nonlinear PDE's involving both time and space derivatives of the metric. Due the coordinate independence of t(87), given one solution another solution can be obtained by making any coordinate transformation, determining the new components of the metric, and then using those components in the original coordinate system. This shows that time evolution of the metric components cannot be fully determined from initial data. When Einstein realized this in 1913 he temporarily concluded that a physical theory could not have full coordinate symmetry. Later he understood that although the metric evolution is not fully determined, any

invariant property—like whether or not two particles collide—is nevertheless fully determined. This underdetermination of  $g_{\mu\nu}$  is brought about by the coordinate freedom of four coordinates, so we can infer that the 10 equations in  $R_{\mu\nu} = 0$  must not all be independent evolution equations. There must be four combinations that involve only first time derivatives, so that their role is only to constrain the initial data, not to determine the evolution. More on this later.

+ **Properties of the Riemann tensor:** established using a local inertial coordinate system.

+ **Number of independent components of the Riemann tensor:** 20, same as the number of second partial derivatives of the metric that cannot in general be set to zero at a point by a coordinate choice.

+ **Linearized field equation:** The linearized Riemann tensor is

$${}^{(1)}R_{\alpha\beta\mu\nu} = 2\partial_{[\alpha}h_{\beta][\mu,\nu]}. \quad (88)$$

The linearized Ricci tensor is

$${}^{(1)}R_{\alpha\beta} = -\frac{1}{2}\square h_{\alpha\beta} + \frac{1}{2}(\partial_{\mu}V_{\nu} + \partial_{\nu}V_{\mu}), \quad V_{\mu} = \partial^{\alpha}h_{\alpha\mu} - \frac{1}{2}\partial_{\mu}h^{\alpha}_{\alpha} \quad (89)$$

+ **Linearized coordinate transformation:** under a linearized coordinate change

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha} \quad (90)$$

the linearized metric changes as

$$h'_{\mu\nu} = h_{\mu\nu} - (\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}). \quad (91)$$

## Tuesday, Nov. 4

+ **Lorentz gauge:** One can choose the coordinates to enforce the gauge condition

$$\partial^{\mu}h_{\mu\nu} - \frac{1}{2}\partial_{\nu}h^{\mu}_{\mu} = 0. \quad (92)$$

To achieve this involves solving the wave equation for  $\xi^{\alpha}$  with a source. The residual "gauge freedom" (linearized coordinate freedom) is that given by solutions to the wave eqn,  $\square\xi^{\alpha} = 0$ . Note that in the linearized theory, the metric perturbation is first order, so metrics used in raising and lowering indices and in derivatives are always the Minkowski metric.

+ **Electromagnetic analogy**: Field variable is the (co)vector potential  $A_\mu$ . The electric and magnetic fields are components of the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (93)$$

This does not look like a tensor at first, since it is constructed with partial derivatives rather than covariant ones. However, the extra term  $-\Gamma^\sigma_{\mu\nu} A_\sigma$  in the covariant derivative  $\nabla_\mu A_\nu$  vanishes upon  $\mu\nu$  antisymmetrization, because the Christoffel symbol is symmetric in  $\mu\nu$ . The spatial components  $F_{ij}$  capture the magnetic field, while the time-space components  $F_{0i}$  capture the electric field. The field strength is invariant under gauge transformations  $A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \lambda$ , because when antisymmetrizing  $\mu\nu$  the mixed partial derivative  $\partial_\mu \partial_\nu \lambda$  vanishes. Note that the linearized Riemann tensor is, similarly, invariant under linearized coordinate transformations.

Maxwell's equations are first order odes, and Lorentz invariant. The ones that relate the fields to the sources correspond, in the absence of sources, to  $\partial^\mu F_{\mu\nu} = 0$ , which in terms of the vector potential reads

$$\partial^\mu F_{\mu\nu} = \square A_\nu - \partial_\nu(\partial^\mu A_\mu) = 0. \quad (94)$$

Using the gauge freedom we can arrange for **Lorentz gauge** to hold,  $\partial^\mu A_\mu = 0$ . In the new gauge we have  $\partial^\mu A'_\mu = \partial^\mu A_\mu - \square \lambda$ , so to set  $\partial^\mu A'_\mu = 0$  we need to solve  $\square \lambda = \partial^\mu A_\mu$ , the wave equation with a source. This can always be solved. We may therefore assume a gauge transformation has been made so that our vector potential satisfies the Lorentz gauge condition, and the field equation (94) is then equivalent to the pair of equations

$$\square A_\mu = 0, \quad \partial^\mu A_\mu = 0. \quad (95)$$

The residual gauge freedom is those gauge parameters  $\lambda$  that satisfy the homogeneous wave equation,  $\square \lambda = 0$ . It's plausible, but not quite obvious (?), that using this residual freedom we can also set  $A'_0 = A_0 - \partial_0 \lambda = 0$ , since  $A_0$  itself is also a solution to the wave equation in Lorentz gauge. Let's check. We define

$$\lambda(t, x^i) = \int_0^t A_0(t', x^i) dt' + \kappa(x^i), \quad (96)$$

for an arbitrary function  $\kappa(x^i)$ , so that  $\partial_0 \lambda = A_0$ , so the gauge condition  $A'_0 = 0$  is satisfied. Now to check whether this  $\lambda$  satisfies the wave equation, we apply the

wave operator:

$$\square\lambda = (-\partial_t^2 + \nabla^2)\lambda \quad (97)$$

$$= -\partial_t A_0(t, x^i) + \int_0^t \nabla^2 A_0(t', x^i) dt' + \nabla^2 \kappa(x^i) \quad (98)$$

$$= -\partial_t A_0(t, x^i) + \int_0^t \partial_{t'}^2 A_0(t', x^i) dt' + \nabla^2 \kappa(x^i) \quad (99)$$

$$= -\partial_t A_0(t, x^i) + \partial_t A_0(t, x^i) - \partial_t A_0(0, x^i) + \nabla^2 \kappa(x^i) \quad (100)$$

$$= -\partial_t A_0(0, x^i) + \nabla^2 \kappa(x^i). \quad (101)$$

This will indeed vanish if we choose  $\kappa$  to be the regular solution to the equation

$$\nabla^2 \kappa(x^i) = \partial_t A_0(0, x^i). \quad (102)$$

Now that we've nailed down the gauge freedom let's look for plane wave solutions:

$$A_\mu = \epsilon_\mu e^{ik_\alpha x^\alpha}. \quad (103)$$

The field equation in Lorentz gauge (95) and the supplementary gauge condition  $A_0 = 0$  are then equivalent to the following conditions on the wave vector  $k_\mu$  and **polarization vector**  $\epsilon_\mu$ :

$$k^2 = 0, \quad k^\mu \epsilon_\mu = 0, \quad \epsilon_0 = 0. \quad (104)$$

So  $k$  can be any null vector, and the polarization vector is a spatial vector orthogonal to the wavevector. This exhibits the two independent linear polarizations for a plane electromagnetic wave. The gauge freedom took us down from four components for each wavevector to just two.

+ In linearized vacuum gravity, the field equation is the vanishing of the linearized Ricci tensor (89), which in Lorentz gauge (92) takes the form

$$\square h_{\mu\nu} = 0, \quad \partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h^\mu{}_\mu = 0 \quad (105)$$

We can use the residual gauge freedom to impose the supplementary conditions

$$h_{0i} = 0, \quad h^\mu{}_\mu = 0. \quad (106)$$

A plane wave solution has the form

$$h_{\mu\nu} = \epsilon_{\mu\nu} e^{ik_\alpha x^\alpha}, \quad (107)$$

where  $\epsilon_{\mu\nu}$  is the **polarization tensor**. The field equation in Lorentz gauge (105) and conditions (106) are then equivalent to

$$k^2 = 0, \quad k^\mu \epsilon_{\mu\nu} = \frac{1}{2} \epsilon k_\nu, \quad \epsilon_{0i} = 0, \quad \epsilon^\mu{}_\mu = 0. \quad (108)$$

The 0 component of the second of these implies  $\epsilon_{00} = 0$ , which together with the third equation implies that  $\epsilon_{\mu\nu}$  is purely spatial. The second equation states that it is transverse to the wave vector, and the last equation implies it is trace free. Therefore this is often called the **transverse-traceless gauge**, or **TT gauge**. There are two linearly independent polarization tensors for each wavevector.

Let's consider the linearized gravitational tidal tensor (84) in TT gauge. Since the Riemann tensor is already  $O(h)$ , all other quantities can be taken to have their Minkowski background values. In particular, the 4-velocity  $T^\alpha$  of the central geodesic can be chosen as just  $(1, 0, 0, 0)$ . Then the tidal tensor has no 0-components, and its spatial part takes the simple form

$$\Upsilon_{ij} = R_{i00j} = \frac{1}{2} h_{ij,00}^{TT}. \quad (109)$$

In a plane gravitational wave, free-falling particles initially at rest experience tidal distortion that is stretching and squeezing in the plane transverse to the wavevector. The stretch and squeeze are opposite, so that as in the general case, the volume of a ball of test particles initially at rest is conserved through second order in time.

## Thursday, Nov. 6

+ **Charge current 4-vector**: A quantity of charge is a scalar. The charge current 4-vector  $j^\mu$  describes the flux of charge through any 3-volume  $dV$ . In a given Lorentz frame, its time component is the charge density  $\rho$  and its spatial components form the charge current density 3-vector  $\mathbf{j}$ . One way to characterize  $j^\mu$  is by the fact that the charge density measured by an observer with 4-velocity  $u^\mu$  is  $-j^\mu u_\mu$ . But  $j^\mu$  also has an interpretation in terms of flux: if the spacetime unit normal to the 3-volume is  $n_\mu$  then the flux—the amount of charge that passes through that 3-volume—is  $j^\mu n_\mu dV_3$ . The direction  $v^\mu$  across the surface is the one for which  $n_\mu v^\mu > 0$ . In the case of spacelike surfaces this means that  $n_\mu = -u_\mu$ , where  $u^\mu$  is a unit normal timelike vector in the direction of the flux.<sup>6</sup> In the timelike case this is the charge passing through a certain area in a certain time interval. In the spacelike case it is the charge in a certain spatial 3-volume at a certain time. Local charge conservation is expressed by the equation

$$\nabla_\mu j^\mu = 0. \quad (110)$$

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<sup>6</sup>For null volumes you can think of a limit in which the product  $n_\mu dV_3$  remains well-defined, while  $n_\mu$  blows up and  $dV_3$  goes to zero.

In local inertial coordinates adapted to a particular observer at a point this reads  $\partial_t \rho + \partial_i j^i = 0$ . Integrating over a small volume and using the divergence theorem, this states that the rate of change of charge in the volume is minus the flux rate of charge outward through the surface of that volume. There is also a global charge conservation law, because the scalar  $j^\mu n_\mu$  integrated over a spatial volume is time independent, thanks to the current conservation (??).

+ **Energy-momentum tensor**: To describe the flux of energy-momentum 4-vector we add an index to the current 4-vector and call it the energy-momentum current tensor  $T^{\mu\nu}$ . There is more than one possible object that goes by this name, because of the ambiguity of adding an identically conserved tensor, but the one that is the source of gravity is unique and is *symmetric*. Since the spatial components also describe the stresses it is also called the stress-energy tensor, the stress tensor, and several other common names. The energy-momentum 4-vector density measured by an observer with 4-velocity  $u^\mu$  is  $\mathcal{P}^\mu = -T^{\mu\nu} u_\nu$ , and the **energy density** is  $T^{\mu\nu} u_\mu u_\nu$ . This fully characterizes  $T^{\mu\nu}$ , but the various components also have other interpretations. In the Lorentz frame of a 4-velocity  $u^\mu$ ,  $T^{00}$  is the energy density  $T^{i0}$  the 3-momentum density,  $T^{0i}$  the energy 3-current, and  $T^{ij}$  the (spatial) stress tensor. Local energy-momentum conservation is expressed by the equation

$$\nabla_\mu T^{\mu\nu} = 0. \quad (111)$$

Unlike with the charge current, there is no global energy-momentum conservation in general, because those quantities are conserved only in the presence of time and space symmetry. The divergence-free condition (111) doesn't lead to a global conservation because  $T^{\mu\nu} n_\mu$  is a vector, and one can't integrate a vector field over a spatial surface in a curved spacetime.

+ **Einstein equation with matter**: Newtonian gravity is governed by the field equation  $\nabla^2 \Phi = 4\pi G \rho_m$ , which states that the trace of the Newtonian tidal tensor [cf. (74)] is proportional to the mass density. In the Newtonian context, this equation is known to fit observations when the free-fall congruence is moves non-relativistically with respect to the source mass. We can derive a relativistic field equation that shares this property by setting the trace of the relativistic tidal tensor  $R_{\mu\nu} u^\mu u^\nu$  (85) equal to  $4\pi G$  times some quantity that for non-relativistic sources reduces to the mass density. The obvious candidate is the energy density  $T_{\mu\nu} u^\mu u^\nu$ , and this leads us to a *candidate field equation*,

$$R_{\mu\nu} = 4\pi G T_{\mu\nu}. \quad (\text{candidate equation \#1}) \quad (112)$$

However there is a fatal flaw in this equation: the **(contracted) Bianchi identity**,

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R, \quad (113)$$

holds for *all* metrics, as a consequence of the definition of the Ricci tensor. Since the stress-energy tensor has vanishing divergence (111), the equation (112) implies that the Ricci scalar  $R$  is *constant* in space and time. The trace of (112) then implies that the trace  $T$  of the stress-energy tensor is also constant, which is a disaster since it is obviously *not* constant in and around the familiar sources of gravity like planets and stars! To escape from this disaster, we can try subtracting  $\frac{1}{2}Rg_{\mu\nu}$  from the left hand side of (112), producing our second candidate equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 4\pi GT_{\mu\nu}. \quad (\text{candidate equation \#2}) \quad (114)$$

Now the left hand side is divergence free, so the divergence of the field equation is consistent with local energy-momentum conservation. However, the left hand side contracted with  $u^\mu u^\nu$  is no longer the trace of the tidal tensor, so we have lost the agreement with the Newtonian limit. To diagnose the damage, we can solve for  $R$  in terms of  $T$ . The trace of (114) implies  $-R = 4\pi GT$ , so (114) is equivalent to

$$R_{\mu\nu} = 4\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \quad (\text{candidate equation \#2}') \quad (115)$$

Now contracting twice with the 4-velocity we find for the trace of the tidal tensor  $4\pi G(T_{\mu\nu}u^\mu u^\nu + \frac{1}{2}T)$ . In the non-relativistic limit this reduces to  $\rho_m - \frac{1}{2}\rho_m = \frac{1}{2}\rho_m$ . To restore the agreement with the Newtonian limit, then, all we need to do is multiply the right hand side by 2, which yields the Einstein field equation,

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}). \quad (\text{Einstein equation}) \quad (116)$$

Equivalently,

$$G_{\mu\nu} = 8\pi GT_{\mu\nu}, \quad (117)$$

where the tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \quad (118)$$

is the **Einstein tensor**.

+ **The source of gravitational attraction** is thus  $T_{\mu\nu}u^\mu u^\nu + \frac{1}{2}T$ . In the case of a perfect fluid at rest in the frame  $u^\mu$  this becomes  $\frac{1}{2}(\rho + 3p)$ , where  $\rho$  and  $p$  are the energy density and pressure. The trace of the tidal tensor is thus given by

$$\Upsilon = R_{\mu\nu}u^\mu u^\nu = 4\pi G(\rho + 3p). \quad (119)$$

We have discovered, just from consistency with local energy-momentum conservation, that not only energy density but pressure is also a source of gravitational attraction. In the case of thermal radiation  $p = \rho/3$ , so the effective source is twice the energy density. A vacuum energy, or cosmological constant, is described by a stress-energy tensor  $T_{\mu\nu} = -\rho_v g_{\mu\nu}$ , with constant energy density  $\rho_v$ . The pressure is then  $p = -\rho_v$ , which is negative if  $\rho_v > 0$ . This produces a negative source  $-2\rho_v$ , hence a **gravitational repulsion!** It was for exactly this reason that Einstein originally introduced the term  $\Lambda g_{\mu\nu}$  into the equation,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (120)$$

The **cosmological constant**  $\Lambda$  is a quantity with the dimensions of an inverse length squared. It is equivalent to a vacuum energy density,  $\Lambda = 8\pi G \rho_v$ . Einstein proposed that it be chosen to balance the average energy density of matter in the universe, to allow for a static universe. That is an unstable balance, not to mention an arbitrary choice, and it is also inconsistent with the observed expansion of the universe. However there is good evidence that in the early universe there was a phase of near-exponential expansion driven by the repulsion of a large vacuum energy that is no longer large today. The present day accelerated expansion may well be driven by a very small vacuum energy, whose origin and magnitude is not yet understood. It goes by the name of **dark energy**. It may just be “left over vacuum energy”.

## Tuesday, Nov. 11 (Details to be filled in later.)

+ Linearized Einstein equation in Lorentz gauge.

+ Analogy with Maxwell’s equations.

+ Bianchi and contracted Bianchi identities.

+ Initial value formulation of Einstein equation.

+ Analogy with Maxwell’s equations.

+ Invariant volume element  $d^4x \sqrt{-g}$ .

+ Einstein-Hilbert action principle for GR.

## Thursday, Nov. 13

+ Comments on the weakness of a gravitational wave amplitude, contrasted with the surprisingly large power flux in the waves, e.g., of amplitude  $10^{-21}$  and frequency 200 Hz that LIGO/VIRGO hopes to observe. See solution to hw9, problem S9-3.

+ **Dimensional analysis and Newton's constant:** The dimensions of the integrand of  $\int d^4x \sqrt{-g} R$  are  $L^4 L^{-2} = L^2$ , so the integral must be multiplied by something with dimensions of action/length<sup>2</sup>. The potential  $GM/r$  is dimensionless (with  $c = 1$ ,  $L = T$ ), so the dimensions of  $1/G$  are energy/length = action/length<sup>2</sup>, just what the doctor ordered. The standard normalization of Newton's constant corresponds to the coefficient  $(1/16\pi G)$  for the Einstein-Hilbert action.

+ Variation of the Einstein-Hilbert action: We need to vary  $\sqrt{-g}$ ,  $g^{\mu\nu}$ , and  $R_{\mu\nu}$ . We already saw that  $\Gamma^\sigma_{\sigma\lambda} = \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\lambda} = (1/\sqrt{-g})\sqrt{g}_{,\lambda}$ . Since the derivative is a variation, we can infer from this that  $\frac{1}{2} g^{\mu\nu} \delta g_{\mu\nu} = (1/\sqrt{-g})\delta\sqrt{-g}$ . Hence

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \quad (121)$$

The last step follows from  $0 = \delta(4) = \delta(g^{\mu\nu} g_{\mu\nu}) = \delta g^{\mu\nu} g_{\mu\nu} + g^{\mu\nu} \delta g_{\mu\nu}$ , which implies  $g_{\mu\nu} \delta g^{\mu\nu} = -g^{\mu\nu} \delta g_{\mu\nu}$ . The tricky part is varying the Ricci tensor. In the end it turns out that this variation contributes only a boundary term, as explained below. Taking this for granted, then with boundary conditions such that the boundary term vanishes, the variation of the EH action is

$$\delta S_{\text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}) \delta g^{\mu\nu}. \quad (122)$$

If this is to vanish for all metric variations, then the Einstein tensor must vanish.

+ The matter action variation,

$$\delta S_{\text{mat}} = \int d^4x \frac{\delta S_{\text{mat}}}{\delta g^{\mu\nu}} \delta g^{\mu\nu}, \quad (123)$$

yields the stress-energy tensor,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{mat}}}{\delta g^{\mu\nu}}. \quad (124)$$

Equation (123) serves to define the notation for the **variational derivative** of the action functional. We will come back to explain why the stress-energy tensor is

defined by (124). Taking this for granted, we conclude that the variation of the full action is given by

$$\delta(S_{\text{EH}} + S_{\text{mat}}) = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi G} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}) - \frac{1}{2} T_{\mu\nu} \right) \delta g^{\mu\nu}. \quad (125)$$

So the variational principle implies the Einstein equation (120).

**+ General curvature variation:** To understand what's happening with the curvature variation it's good to consider the variation of the curvature of any connection:

$$\delta\Omega_{\mu\nu} = \delta(2\partial_{[\mu}B_{\nu]} + 2B_{[\mu}B_{\nu]}) = 2D_{[\mu}B_{\nu]}. \quad (126)$$

In the last expression,  $D_\mu$  is the gauge-covariant derivative introduced in (55). Here's a really slick way to see the last step: at any point  $p$ , choose a gauge in which  $B_\mu$  vanishes at  $p$ . In that gauge, the variation evaluated at  $p$  is just  $2\partial_{[\mu}\delta B_{\nu]}$ , which in that gauge is also equal to  $2D_{[\mu}\delta B_{\nu]}$ . Now the latter expression is covariant, although you might think not because the connection transforms inhomogeneously, as in (57). But when the variation of (57) is taken, the inhomogeneous term does not contribute, so we have  $\delta B'_\mu = g\delta B_\mu g^{-1}$ . Since  $2D_{[\mu}\delta B_{\nu]}$  transforms the same way as  $\delta\Omega_{\mu\nu}$ , and they are equal at  $p$  in one gauge, they are equal at  $p$  in all gauges. *QED*. Alternatively, you can just vary the  $BB$  term in (126), and recognize the extra terms as the  $B$ -terms in the expression for  $2D_{[\mu}\delta B_{\nu]}$ .

**+ Ricci tensor variation in the action** We can apply the above general result to the Riemann tensor, considering that the metric variation induces some variation of the Levi-Civita connection. We thus have

$$\delta R^\sigma{}_\tau{}_{\mu\nu} = 2\nabla_{[\mu}\delta\Gamma^\sigma{}_{\nu]\tau}, \quad \delta R_{\mu\nu} = 2\nabla_{[\sigma}\delta\Gamma^\sigma{}_{\mu]\nu}, \quad (127)$$

so the variation of the Einstein-Hilbert action includes the contribution

$$\nabla_{[\sigma}(2\sqrt{-g}g^{\mu\nu}\delta\Gamma^\sigma{}_{\mu]\nu}) = \nabla_\sigma(2\sqrt{-g}g^{\nu[\mu}\delta\Gamma^\sigma]{}_{\mu\nu}) \quad (128)$$

$$= \partial_\sigma(2\sqrt{-g}g^{\nu[\mu}\delta\Gamma^\sigma]{}_{\mu\nu}). \quad (129)$$

In the first step I traded the antisymmetrizer on the lower  $\sigma\mu$  pair for the antisymmetrizer on the upper pair with which this lower pair is contracted. This is valid because contraction an index pair with an antisymmetric index pair projects out the antisymmetric part of the former. In the last step I used the result from HW8, Problem S8-4, that the covariant divergence of  $\sqrt{-g}$  times a vector is equal to the coordinate divergence of the same quantity. Being a coordinate divergence, integration over it gives just a boundary term. Notice that we don't need to work out

the expression for  $\delta\Gamma_{\mu\nu}^\sigma$  in terms of  $\delta g^{\mu\nu}$  in order to conclude that this term contribute to the variation when the boundary term vanishes.

**+ Electromagnetic field action**

$$S_A = -\frac{1}{4} \int d^4x \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}. \quad (130)$$

**+ Scalar field action:**

$$S_\varphi = - \int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \right). \quad (131)$$

Note the Lagrangian has the form kinetic minus potential energy. In local inertial coordinates, the kinetic energy density is  $\frac{1}{2}(\partial_0\varphi)^2$  while the potential energy density is  $\frac{1}{2}(\partial_i\varphi)^2 + V(\varphi)$ , the gradient energy plus the “potential”. For instance, a free massive scalar has a quadratic potential,  $V(\varphi) = \frac{1}{2}m^2\varphi^2$ . A term proportional to  $R\varphi^2$  is called a **non-minimal coupling** term. We worked out the stress-energy tensor for the scalar, and it is also in one of the problems of HW11.

## Tuesday, Nov. 18

**+ Ingredients of the Universe:** Most of this is in Hartle Ch. 17. Ordinary matter, radiation, dark matter, and dark energy. A few additional comments: 1) The wavelength of the typical thermal radiation in the CMB is about 1 mm, so there is about 1 photon per cubic mm. On the other hand there is about one proton per cubic meter. So there are about  $10^9$  photons per baryon. 2) I discussed a bit more about the possibilities for dark matter. 3) The “dark energy” plays two roles: a) this energy density is required for consistency with the observed flatness of the spatial geometry of the universe, and b) the associated negative pressure is required to produce the accelerated expansion that has been observed. The simplest possibility is that this dark energy is just a vacuum energy, left over after the various phase transitions that have happened in the early universe. 4) Energy conservation together with a constant energy density implies  $p = -\rho$ , the equation of state of vacuum energy: if a volume of space expands by  $dV$ , the work done by the “fluid” in that volume is  $p dV$ . This should be equal to minus the energy change of that fluid,  $-\rho dV$ , therefore  $p = -\rho$ . This also follows from conservation of the stress tensor.

**+ Anthropic prediction for vacuum energy:** Weinberg (I’m not sure if he was the first to suggest this) postulated that a small value of the cosmological can be explained “anthropically”, because if it were much larger the universe would have

gone into accelerated expansion before there was time for structure to form, and if it were negative the universe would have recollapsed (I think — should check this). (<http://journals.aps.org/rmp/abstract/10.1103/RevModPhys.61.1>) So, given that we are here, only values near zero should be seen. And this “predicts” that a nonzero value would be seen, which is exactly what happened in 1998, after Weinberg’s 1989 paper.

+ **Discovery of expansion of the universe:** 1915, GR is finished. 1917, Einstein introduces the static universe model with cosmological constant, and de Sitter introduces his solution. Friedmann in 1922 finds expanding solutions with dust, and no cosmological constant (I think). Einstein publishes a note saying there is an error because energy is not conserved, but it is Einstein who is mistaken (he misinterpreted the consequence of conservation of the stress tensor — he seems to have forgotten the factor of  $\sqrt{-g}$  in the conservation law  $\sqrt{-g}T^{00} = \text{constant}$ . An associate of Friedmann’s catches up with Einstein in Paris and explains it to him. Einstein publishes a retraction, but concludes with a remark that nevertheless the solution has no physical interest(!))

Einstein’s static model is (obviously, I would say) unstable. Eddington pointed this out in 1930, after reading Lemaître’s paper, where he says it is demonstrated but not explicitly commented upon — Eddington’s paper:  
<http://adsabs.harvard.edu/abs/1930MNRAS..90..668E>.

The trailblazing (but initially ignored) 1927 paper by Georges Lemaître explained “everything” (it is linked at the Supplements page of the course website). He interpreted the redshift as due to cosmic expansion, found the relevant solutions to the Einstein equation with matter and a cosmological constant, and also used the best data at the time to estimate the current expansion rate (a.k.a. **Hubble constant**) using the redshifts measured by Strömberg and the magnitudes measured by Hubble, from which distances can be inferred. The value he found was 625 km/s/Mpc, fairly close to the value 500 km/s/Mpc found by Hubble two years later in 1929 using better distance measurements. Hubble’s value was about 7 times as large as today’s best value,  $67.80 \pm 0.77$  km/s/Mpc, because of several factors related to incorrect inference of distance from magnitudes. That means that without the cosmological constant the universe would have been around 7 times younger, which would have been too young to accommodate the age of the earth and the oldest stars. But with the cosmological constant the expansion is accelerating, so it is larger today than it was in the past, so that the age of the universe can be longer. So Lemaître believed that a cosmological constant was needed for agreement with observations. In fact that is true today, but for much smaller values of the expansion rate and cosmological constant.

+ **Metric of isotropic spacetime:** Assuming isotropy about every point, the isotropic observer world lines must be orthogonal to 3d spatial hypersurfaces that are homogeneous and isotropic, hence maximally symmetric. The geometry of these 3d surfaces is therefore either  $S^3$ ,  $R^3$ , or  $H^3$ . These surfaces can be labelled by the proper time  $t$  along the isotropic observer worldlines. The spacetime line element thus has the form:

$$ds^2 = -dt^2 + a^2(t) \left( d\chi^2 + \Phi^2(\chi) d\Omega^2 \right), \quad \Phi(\chi) = \{\sin \chi, \chi, \sinh \chi\} \quad (132)$$

or, using a different radial coordinate,

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \quad k = \{1, 0, -1\}. \quad (133)$$

In the  $S^3$  case ( $k = 1$ ) the second coordinate system covers only half the  $S^3$ . Note: we could also imagine quotients of these spaces, obtained by identification modulo a discrete subgroup of the symmetry group that acts without fixed points.

+ **Cosmological redshift:** Consider two successive wave crests separated by the coordinate difference  $\Delta\chi$ . By virtue of the symmetry of the spacetime, as they propagate to the future the crests will maintain the same coordinate separation on each spatial slice  $\Sigma_t$ . The proper wavelength on these slices is thus time dependent,  $\lambda(t) = a(t)\Delta\chi$ . If the wave is emitted at time  $t_e$  and observed at time  $t_o$  the ratio of the wavelengths is  $\lambda_o/\lambda_e = a_o/a_e$ . The redshift  $z$  is defined by

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e} = \frac{a_o - a_e}{a_e} = \left( \frac{\dot{a}}{a} \right)_o (t_o - t_e) + O[(t_o - t_e)^2]. \quad (134)$$

The Hubble “constant”  $H$  is defined by

$$H = \left( \frac{\dot{a}}{a} \right)_o. \quad (135)$$

If the redshift is interpreted as a Doppler shift, then  $1 + z = [(1 + v/c)/(1 - v/c)]^{1/2} = 1 + v/c + O(v^2/c^2)$ , so to lowest order we have  $z = v/c$ . In terms of this  $v$ , the cosmological redshift formula (134) reads  $v = Hd$ , where  $d = c(t_o - t_e)$  is the distance the light has travelled. The linear relation between  $z$  and  $d$  is called the “Hubble law”. It could be explained in a flat spacetime setting if the galaxies all started out at the same point in space and time in an explosion, traveling in a given time a distance proportional to their velocity. This interpretation can’t be maintained once the velocity approaches the speed of light, and the cosmological

redshift only agrees with the recessional velocity picture in a sufficiently small neighborhood.

Student question: how can the redshift be explained both as due to expansion of the universe and as due to velocity of recession? Doesn't the expansion stretch the wavelength as the wave propagates, whereas recession is a kinematic effect at emitter and receiver? My answer is that first of all, in a spacetime region small compared to the radius of curvature, which here is the inverse expansion rate  $a/\dot{a}$ , it must be correct to describe the physics as in flat spacetime. Then cosmological isotropic observers at greater distances must have greater relative recessional velocity, so as the wave passes from one observer to another more remote one, it is progressively observed as more and more redshifted.

## Thursday, Nov. 20

+ **Parsec:** Distance at which 1AU subtends and angle of 1”.

+ **Cosmological time dilation:** Last time we derived the redshift by using spatial translation invariance to argue that the wavelength in co-moving coordinates is constant, so the proper wavelength is proportional to the scale factor  $a(t)$ . Another derivation, which reveals a further significance, goes as follows. A null curve satisfies  $ds^2 = 0$ , so in propagating from  $\chi_1$  to  $\chi_2$  we have the integral  $\int_{\chi_e}^{\chi_o} \delta\chi = \int_{t_e}^{t_o} dt/a(t)$ . If the lower limit is replaced by  $t_e + \delta t_e$ , the upper limit is replaced by  $t_o + \delta t_o$ , such that the integral is unchanged since the  $\chi$  integral is the same. If  $\delta t_e$  and  $\delta t_o$  are small compared to the time over which  $a(t)$  changes appreciably, then we must have  $\delta t_e/a_e = \delta t_o/a_o$ , hence

$$\delta t_o/\delta t_e = a_o/a_e. \quad (136)$$

This relation applies to the period of an electromagnetic wave, or to any other time interval observed by signals traveling at the speed of light. For instance, the decay of the light curve of a supernova takes an observed time that is longer than its intrinsic decay time by the redshift factor  $a_o/a_e$ .

+ The **dynamics of the cosmological metric** is governed by the Einstein equation together with the equations of motion of the matter. When we treat the matter as a perfect fluid with stress-energy tensor

$$T_{ab} = \rho u_a u_b + p(g_{ab} + u_a u_b), \quad (137)$$

the conservation equation  $\nabla_a T^{ab} = 0$  plays the role of the matter equation of motion. In a homogeneous, isotropic geometry the fluid 4-velocity  $u^a$  coincides

with that of the isotropic frame, and the divergence of  $T_{ab}$  is proportional to  $u^a$ , so the conservation equation has only one nontrivial component,  $\nabla_a(\rho u^a) + p\nabla_a u^a = 0$ . Using  $\sqrt{-g}\nabla_a V^a = \partial_\alpha(\sqrt{-g}V^\alpha)$ , the conservation equation can be expressed as  $\partial_t(\rho a^3) + p\partial_t(a^3) = 0$ . This is just the first law of thermodynamics with no heat flux,

$$dE + p dV = 0, \quad (138)$$

where  $E = \rho a^3 \mathcal{V}$  is the energy in a comoving region with a proper volume  $V = a^3 \mathcal{V}$  corresponding to a volume  $\mathcal{V}$  as defined by the fixed spatial metric. Equivalently,

$$\dot{\rho} + 3(\rho + p)(\dot{a}/a) = 0. \quad (139)$$

Given the equation of state  $p = p(\rho)$ , this can be integrated to find  $\rho$  as a function of  $a$ . Examples:

matter	$p = 0$	$\rho \sim a^{-3}$
radiation	$p = \rho/3$	$\rho \sim a^{-4}$
vacuum	$p = -\rho$	$\rho = \text{constant}$
stiff fluid	$p = \rho$	$\rho \sim a^{-6}$
spatial curvature	$p = -\rho/3$	$\rho \sim a^{-2}$

A homogeneous scalar field with no potential or mass behaves as a “stiff fluid”.

+ Different components of the energy density thus dominate at different times.  $z$  at matter-radiation equality is about 3600, at last scattering is about 1100, and at vacuum-matter equality is about 0.67. At last scattering the temperature was therefore 1100 times the CMB temperature today of 2.7K, hence was about 3000K. This corresponds to an energy of about 0.3 eV, and a peak of the thermal distribution of frequency around 0.8 eV. This is less than 10% of the ionization energy 13.6 eV of hydrogen, but the thermal distribution has enough higher energy photons above this temperature to keep the hydrogen ionized.

+ The Einstein tensor has the same structure as the perfect fluid stress-energy tensor (137), so there are just two independent components of the Einstein equation, the  $tt$  component and the  $ii$  component. As explained above, the  $tt$  equation is involves only first time derivatives — it is an initial value constraint equation — and it is automatically preserved in time if the other field equations hold and the stress-energy tensor is conserved. In the present case, conversely, if we impose the  $tt$  equation for all time, along with matter energy conservation, then the other, second order Einstein equation is automatically satisfied. This is like using

the conserved energy for a particle in one-dimension instead of the second order equation of motion.

For the cosmological metric, we have

$$G_{tt} = 3 \left( \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right), \quad (140)$$

hence the  $tt$  equation yields

$$\dot{a}^2 - \frac{8\pi G}{3} \rho a^2 = -k, \quad (141)$$

which is known as the **Friedman equation**. The time derivative of this equation, together with (139) yields

$$3\ddot{a}/a = 4\pi G(\rho + 3p) \quad (142)$$

Comparing with (119) for the trace of the tidal tensor we see that evidently  $R_{tt} = 3\ddot{a}/a$ . In fact, for a metric of this simple form it is easy to see this directly.<sup>7</sup>

One can also arrive at the Friedman equation directly from the action. Evaluating the Ricci scalar for the metric

$$ds^2 = -N^2(t)dt^2 + a^2(t)q_{ij}dx^i dx^j, \quad (143)$$

where  $q_{ij}$  is the metric of a maximally symmetric 3d space,

$$q_{ij}dx^i dx^j = \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (144)$$

with  $k = 0, \pm 1$ , the Einstein-Hilbert action takes the form

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R = \frac{6V_3}{16\pi G} \int dt (kNa - a\dot{a}^2/N), \quad (145)$$

where  $V_3 = \int d^3x \sqrt{q}$  (which may be infinite). As for the matter action, we have

$$\delta S_{\text{m}}/\delta N(t) = -\sqrt{-g}T_{tt} = -Na^3\rho. \quad (146)$$

Varying the full action with respect to  $N(t)$ , and then choosing the coordinate gauge  $N(t) = 1$ , yields directly the Friedman equation.

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<sup>7</sup>The connecting vector of the one-parameter family of isotropic geodesics,  $x^\alpha(\chi, t) = t\delta_t^\alpha + \chi\delta_\chi^\alpha$ , is  $S^\alpha = a\hat{S}^\alpha$ , where  $\hat{S}^\alpha$  is a unit spacelike vector, orthogonal to and parallel transported along the geodesics. Thus we have  $(u \cdot \nabla)(u \cdot \nabla)S^\alpha = (\ddot{a}/a)S^\alpha$ , so the trace of the isotropic tidal tensor is  $3\ddot{a}/a$ .