Supplement—Phys 675—Fall 2007 Prof. Ted Jacobson
Room 4115, (301)405-6020
jacobson@umd.edu

Non-affinely parametrized geodesic equation

Consider first flat spacetime, in Minkowski coordinates. The geodesics are straight lines, which can be parametrized as

$$x^\mu(\sigma) = x^\mu(0) + \dot{x}^\mu\,\sigma. \tag{1}$$

This satisfies the equation $d^2x^\mu/d\sigma^2 = 0$, which agrees with the affinely parameterized geodesic equation in the case $g_{\mu\nu} = \eta_{\mu\nu}$. Thus σ is an affine parameter for these geodesics. If λ is any other parameter for the same curve, then

$$\frac{d^2x^\mu}{d\lambda^2} = \frac{d}{d\lambda}\left(\frac{dx^\mu}{d\sigma}\frac{d\sigma}{d\lambda}\right) \tag{2}$$

$$= \frac{d^2x^\mu}{d\sigma^2}\left(\frac{d\sigma}{d\lambda}\right)^2 + \frac{dx^\mu}{d\sigma}\frac{d^2\sigma}{d\lambda^2} \tag{3}$$

$$= f(\lambda)\frac{dx^\mu}{d\lambda}, \tag{4}$$

where

$$f(\lambda) = \frac{d^2\sigma/d\lambda^2}{d\sigma/d\lambda}. \tag{5}$$

That is, in this flat spacetime example, a non-affinely parametrized geodesic has a nonzero coordinate acceleration that is proportional to its velocity. Since the acceleration is proportional to the velocity, it can be eliminated by an appropriate change of parameter, in this case from λ to σ.

In arbitrary coordinates for flat spacetime, or in a curved spacetime, the non-affinely parametrized geodesic equation takes the form

$$\frac{d}{d\lambda}(g_{\alpha\nu}\dot{x}^\nu) - \tfrac{1}{2}g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu = f(\lambda)g_{\alpha\nu}\dot{x}^\nu, \tag{6}$$

where $f(\lambda)$ can be any nonzero function. In local inertial coordinates at a point p this becomes

$$\eta_{\alpha\nu}\ddot{x}^\nu|_p = f(\lambda)\eta_{\alpha\nu}\dot{x}^\nu|_p, \tag{7}$$

which (since $\eta_{\mu\nu}$ is invertible) is equivalent to the statement that the acceleration is proportional to the velocity, as above.