Supplement for Phys675, Fall 2004, T. Jacobson

Determinant = volume ratio

The determinant of a linear transformation in n-dimensions is the factor by which the transformation changes n-volumes. To demonstrate this we need two facts:

1. The volume of a parallelopiped formed by n vectors $V_1 \dots V_n$ in an n-dimensional space is given by

$$\text{vol}(V_1, \dots, V_n) = \epsilon_{a_1 \dots a_n} V_1^{a_1} \dots V_n^{a_n}, \tag{1}$$

   where $\epsilon_{a_1 \dots a_n}$ is the volume element, equivalently, its components are the alternating symbol in an orthonormal basis.

2. The determinant of a linear transformation L satisfies

$$L^{a_1}_{b_1} \dots L^{a_n}_{b_n} \epsilon_{a_1 \dots a_n} = \det L \, \epsilon_{b_1 \dots b_n}. \tag{2}$$

Now, compute the volume of the parallelopiped formed by the n vectors $LV_1 \dots LV_n$:

$$\begin{aligned}
\text{vol}(LV_1, \dots, LV_n) &= \epsilon_{a_1 \dots a_n} (L^{a_1}_{b_1} V_1^{b_1}) \dots (L^{a_n}_{b_n} V_n^{b_n}) && (3) \\
&= \det L \, \epsilon_{b_1 \dots b_n} V_1^{b_1} \dots V_n^{b_n} && (4) \\
&= \det L \, \text{vol}(V_1, \dots, V_n). && (5)
\end{aligned}$$