$\rightarrow T_{\alpha\beta} = (\rho + p) u_\alpha u_\beta + p g_{\alpha\beta}$

· vacuum: $T_{\alpha\beta} = -\rho_v\, g_{\alpha\beta}$, $\rho_v =$ vac. energy density

$= T_{\alpha\beta} u^\alpha u^\beta$

$=$ const. ← How do I know this?

$T_{\alpha\beta} = \frac{1}{4\pi}\left[F_{\alpha\mu} F_\beta{}^\mu - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right]$

$$\left.\begin{aligned} \overset{E}{\Phi}{}_\alpha{}^\alpha &= -4\pi G\, T_{\mu\nu} u^\mu u^\nu \\ &= -R_{\mu\nu} u^\mu u^\nu \end{aligned}\right] \Rightarrow \boxed{R_{\mu\nu} = 4\pi G\, T_{\mu\nu}} \quad \text{Guess \# 1.}$$

Does not conserve energy consistently.

$\vec\nabla \cdot \vec E = 4\pi\rho$

$\vec\nabla \cdot \vec B = 0$

$\vec\nabla \times \vec E = -\partial_t \vec B$

$\vec\nabla \times \vec B = \vec J 4\pi$

But Maxwell knew $\vec\nabla \cdot (\vec\nabla \times \vec f) = 0$
(mix partials commute, reason why anything is true).
So $\vec\nabla \cdot \vec J = 0$, but this is not true, only true in steady state. In general, $\vec\nabla \cdot \vec J = -\partial_t \rho$

$\vec\nabla \cdot \vec E = 4\pi\rho$, $\vec\nabla \cdot \vec J = -\frac{1}{4\pi} \vec\nabla \cdot (\partial_t \vec E)$.

$\vec\nabla \times \vec B = 4\pi \vec J + \partial_t \vec E$

Relativistic notation for charge conservation & ME.

$F_{[\alpha\beta,\gamma]} = 0$ identity if $F_{\alpha\beta} = 2\, \partial_{[\alpha} A_{\beta]}$

$\partial_\alpha F^{\alpha\beta} = 4\pi j^\beta$ ⇒ charge conservation $\partial_\beta \partial_\alpha F^{\alpha\beta} = 0$

⇒ consistency of $F^{\alpha\beta}$ w/ charge cons.

Energy-momentum conservation $\nabla_\alpha T^{\alpha\beta} = 0$, local equation; not global total

Then it'd better be true that $\nabla_\alpha R^{\alpha\beta} = 0$, but it is not!
But I can not modify it in a way that it messes up the Newtonian limit.

$\nabla^\alpha \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) \equiv 0$, Bianchi Identity.

$\nabla^\alpha G_{\alpha\beta} = 0$ for any metric.

Answer: $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G \, T_{\alpha\beta}$

$$R - \frac{1}{2} \cdot 4 \cdot R = 8\pi G T$$

$$R = -8\pi G T$$

I knew $R_{\alpha\beta} = 4\pi G T_{\alpha\beta}$

$$R_{\alpha\beta} = 8\pi G T_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} (-8\pi G T)$$

$$= 4\pi G \left[2 T_{\alpha\beta} - g_{\alpha\beta} T \right]$$

So we need to compare $T_{\alpha\beta} = 2 T_{\alpha\beta} - g_{\alpha\beta} T$

for non-relativistic $T_{\alpha\beta} = \left(\begin{array}{c|c} \rho & 0 \\ \hline 0 & 0 \end{array}\right)$, so $T = -\rho$.

So $2 T_{00} - g_{00} T = 2\rho - (-1)(-\rho) = \rho = T_{00}$

Dec 9^{th}

Nature of Einstein Eqs. Some of them are constraint eqs.

$\mathrm{Tr} \, \overset{N}{\Phi}{}_i{}^i = -4\pi G \rho_{mass}$

$\mathrm{Tr} \to -R_{\mu\nu} u^\mu u^\nu$,

$$\rho_{mass} \to \rho_{ENERGY} \to \begin{cases} T_{ab} u^a u^b \\ -T = -T_{ab} g^{ab} \end{cases}$$

These two are close, so in guessing, some linear combination. Energy & momentum conservation tells us what appears.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Conservation of energy $\nabla^\mu T_{\mu\nu} = 0$.

The field equation does not do this for me, with only $R_{\mu\nu}$.

$\nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R \leftarrow$ contracted w/ Bianchi

$$R - \frac{1}{2} R \cdot 4 = 8\pi G T$$

$$R = -8\pi G T$$

$$R_{\mu\nu} = 8\pi G T_{\mu\nu} + \frac{1}{2}(-8\pi G T) g_{\mu\nu}$$

$$= 4\pi G (2 T_{\mu\nu} - T g_{\mu\nu})$$

$T_{\mu\nu}$ for a perfect fluid ; $T_{\mu\nu} = \rho u_\mu u_\nu + \overbrace{p (g_{\mu\nu} + u_\mu u_\nu)}^{\text{spatial}}$

$$\left(\begin{array}{c|ccc} \rho & & & \\ \hline & p & & \\ & & p & \\ & & & p \end{array}\right)$$

$$T = -\rho + 3p$$

energy density $\rightarrow$ $(2 T_{\mu\nu} - T g_{\mu\nu}) u^\mu u^\nu = 2\rho + (-\rho + 3p) = \rho + 3p$

$$T_{\alpha\beta} = \frac{1}{4\pi}\left[F_{\alpha\mu} F_\beta{}^\mu - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu}\right]$$

$$T = \frac{1}{4\pi}\left[F_{\alpha\mu} F^{\alpha\mu} - \frac{1}{4}\cdot 4\, F_{\mu\nu} F^{\mu\nu}\right] = 0 .$$

Taking trace , $-\rho + 3p$, that should be 0, tells us that $p = \frac{1}{3}\rho$ for thermal radiation.

What it means for "source of gravity of attraction"

How does this mean source of gravity? $\longrightarrow$ $(2T_{\mu\nu} - T g_{\mu\nu}) u^\mu u^\nu = \rho + 3p = 2\rho$

We are on a roll! So what about "vacuum energy" ?

$$T_{\mu\nu} = -\rho_v\, g_{\mu\nu}$$

- Locally Lorentz Inv't , $-T_{\mu\nu} u^\mu u^\nu = \rho$:

But here $p = -\rho$ from perfect fluid form $T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu)$

So the source of attraction $= \rho + 3p = -2\rho_v$.

This is "anti-gravity!"

one technical thing : contracted Bianchi identity

Bianchi identity $\quad R_{\mu\nu[\alpha\beta;\gamma]} = 0$

In a local inertial coordinate system

$$R_{\mu\nu[\alpha\beta,\gamma]}$$

$$= \frac{1}{2}\partial_{[\mu} g_{\nu][\alpha,\beta],\gamma]} \quad \leftarrow \text{holy God!}$$

anti-symmetrizer on two partials.

der of der of metric , it could be a problem.

$R = \partial\Gamma + \Gamma\Gamma$

$R_{,\alpha} = \partial\partial\Gamma + \partial\Gamma\,\Gamma +$

$$R_{\mu\nu\alpha\beta;\gamma} + R_{\mu\nu\gamma\alpha;\beta} + R_{\mu\nu\beta\gamma;\alpha} = 0$$

$\mu \leftrightarrow \alpha$
$\nu \leftrightarrow \beta$

$$R_{;\gamma} + - R_{\nu\gamma;}{}^{\nu} - R_{\mu\gamma;}{}^{\mu} = 0$$

$$R_{;\gamma} = 2\, R_{\mu\gamma;}{}^{\mu} \quad \rightarrow 2\nabla^\mu R_{\mu\nu} = \nabla_\nu R .$$

contracted Bianchi.

of Eq and # of Things to Be Determined

$G_{\mu\nu} = 8\pi\, T_{\mu\nu}$, $g_{\mu\nu}$ has 10 quantities , 4 coordinate fcn freedom.

4/10 of E.E. must not specify independent

Indeed, $G_{0\mu} = 8\pi T_{0\mu}$ involves no 2^{nd} deri of $g_{\mu\nu}$.

$$\nabla_\mu G^{\mu\nu} = 0 \rightarrow \partial_\mu G^{\mu\nu} + \Gamma G = 0$$

$$\rightarrow \underbrace{\partial_0}_{\partial_t} \underbrace{G^{0\nu}}_{?} + \underbrace{\underbrace{\partial_i}_{\partial_i} \underbrace{G^{i\nu}}_{\partial^2}}_{\text{no } \partial_t^3} + \underbrace{\underbrace{\Gamma}_{\partial} \underbrace{G}_{\partial^2}}_{\text{no } \partial^3} = 0$$

Initial Data

$g_{\mu\nu} \big| \quad g_{\mu\nu,\alpha} \big|$

So this must not have ∂_t^2 in it !
because the entire equation adds to zero.

(How come "no ∂_t^2" $\Rightarrow$ "does not evolve" ?)

$\partial_\mu F^{\mu\nu} = 4\pi J^\nu$, must not determine all 4 components! Gauge Inv.

$$\partial_\mu F^{\mu 0} = \partial_0 F^{00} + \underbrace{\partial_i F^{i0}}_{\text{at most } \partial_t}$$

Gauss' Law $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$, does not evolve, but constrains initial data. This equation has only one time derivative of A^μ. Since $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ is independent of time, so this contraint is satisfied for all times.

[Take $\partial_0 (G_{0\mu} - 8\pi T_{0\mu})$ using $\nabla^\mu G_{\mu\nu} = 0$]

$$\partial_0 G_{0\mu} - 8\pi\, \partial_0 T_{0\mu}$$
$$= -\partial_i G_{i\nu} - \Gamma G + 8\pi\, \partial_i T_{i\mu}$$

For numerical GR, I need initial data that satisfy $G_{0\mu} = 8\pi T_{0\mu}$.

The Action Principle

Use Lagrangian, vary paths, variation of action is 0 $\Rightarrow$ field eq.

Einstein-Hilbert Action

$$S[g] = \underbrace{\int d^4x \sqrt{-\det g}}_{\text{proper volume}}\; R \,\frac{1}{16\pi G}, \quad \underbrace{\frac{\delta S}{\delta g^{\mu\nu}} \propto R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}}$$

$$S[g] = \int d^4x \sqrt{-\det g}\; R + S\left[\begin{matrix}\text{matter} \\ \text{fields}\end{matrix}, A_\mu\right] ;$$

$\frac{\delta R_{\mu\nu}}{\delta g_{\alpha\beta}} \Rightarrow$ pure divergence.

"God is merciful!"

$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \Rightarrow$ Einstein Eq.