This problem is borrowed from a course taught by Hitoshi Murayama.

1. Bhabha scattering. The scattering of an electron and a positron is called Bhabha scattering. We compute the scattering cross section using the following four-momenta

$$\begin{aligned} p &= E(1,0,0,1) \quad (1)\\ \bar{p} &= E(1,0,0,-1) \quad (2)\\ k &= E(1,\sin\theta,0,\cos\theta) \quad (3)\\ \bar{k} &= E(1,-\sin\theta,0,-\cos\theta) \quad (4) \end{aligned}$$

for the initial electron, initial positron, final electron, and final positron, respectively. We neglect the masses of electron and positron. There are two Feynman diagrams for this process, one is the annihilation of electron and positron into a virtual photon which creates an electron positron pair, the other is the exchange of a virtual photon between an electron and a positron. The Mandelstam variables are defined as usual, $s = (p+\bar{p})^2 = (k+\bar{k})^2$, $t = (p-k)^2 = (\bar{p}-\bar{k})^2$, and $u = (p-\bar{k})^2 = (\bar{p}-k)^2$.

(a) Using the LSZ reduction formula, the matrix element is related to the correlation function of the Heisenberg operators,

$$\langle \Omega | T e(y)\bar{e}(x)\bar{e}(\bar{y})e(\bar{x})|\Omega\rangle, \quad (5)$$

where the space time coordinates x, $\bar{x}$, y, $\bar{y}$ refer to the positions of the initial electron, initial positron, final electron, and final positron, respectively. The operators $e(y)$, $\bar{e}(x)$, $\bar{e}(\bar{y})$, $e(\bar{x})$ are multiplied by the spinors $\bar{u}(k)$, $u(p)$, $v(\bar{k})$, and $\bar{v}(\bar{p})$, respectively, after stripping off the propagator piece. Consider a perturbation at $O(e^2)$ where e is the QED coupling constant. Convince yourself that there are two Feynman amplitudes

$$\begin{aligned} i\mathcal{M} &= \bar{u}(k)(-ie\gamma^\mu)u(p)\frac{-ig_{\mu\nu}}{t}\bar{v}(\bar{p})(-ie\gamma^\nu)v(\bar{k}) \\ &\quad -\bar{u}(k)(-ie\gamma^\mu)v(\bar{k})\frac{-ig_{\mu\nu}}{s}\bar{v}(\bar{p})(-ie\gamma^\nu)u(p) \quad (6) \end{aligned}$$

with a relative *minus* sign. For later convenience, we define

$$\begin{aligned} \mathcal{A}_1 &= \bar{u}(k)\gamma^\mu u(p)\bar{v}(\bar{p})\gamma_\mu v(\bar{k}), \quad (7)\\ \mathcal{A}_2 &= \bar{u}(k)\gamma^\mu v(\bar{k})\bar{v}(\bar{p})\gamma_\mu u(p) \quad (8) \end{aligned}$$

such that

$$i\mathcal{M} = \frac{ie^2}{t}\mathcal{A}_1 - \frac{ie^2}{s}\mathcal{A}_2\,. \quad (9)$$

(b) We evaluate the spin-summed squared amplitude of the first term. Show that

$$\sum_{\text{helicities}} |\mathcal{A}_1|^2 = \mathrm{Tr}(\not{k}\gamma^\mu \not{p}\gamma^\nu)\mathrm{Tr}(\not{\bar{p}}\gamma_\mu \not{\bar{k}}\gamma_\nu) = 8(u^2+s^2). \tag{10}$$

Note that this is exactly the same expression for the $e\mu$ scattering process in the massless muon limit.

(c) We evaluate the spin-summed squared amplitude of the second term. Show that

$$\sum_{\text{helicities}} |\mathcal{A}_2|^2 = \mathrm{Tr}(\not{k}\gamma^\mu \not{\bar{k}}\gamma^\nu)\mathrm{Tr}(\not{\bar{p}}\gamma_\mu \not{p}\gamma_\nu) = 8(u^2+t^2). \tag{11}$$

Note that this is related to the first term by the *crossing*, by the substitution $k \to -\bar{p}$ and $\bar{p} \to -k$. Note that this is exactly the same expression for the $e^+e^- \to \mu^+\mu^-$ in the massless limit.

(d) There are two cross terms in the squared amplitude. Show that

$$\sum_{\text{helicities}} \mathcal{A}_1\mathcal{A}_2^* = \mathrm{Tr}(\not{k}\gamma^\mu \not{p}\gamma_\nu \not{\bar{p}}\gamma_\mu \not{\bar{k}}\gamma^\nu) = -8u^2. \tag{12}$$

Here, the identities $\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}$, and $\gamma^\mu \not{a}\not{b}\gamma_\mu = 4(a\cdot b)$ are useful. Since the final expression is real, the other cross term, $\mathcal{A}_2\mathcal{A}_1^*$ is the same as this one.

(e) Show that the differential cross section for unpolarized beams is given by

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left[\frac{u^2+s^2}{t^2} + \frac{u^2+t^2}{s^2} + \frac{2u^2}{st}\right] \tag{13}$$

(f) Using the four-momenta in the center-of-momentum frame given above, depict the $\cos\theta$ dependence of the first term, second term, third term, and the sum in the squared bracket. Note that the first term is far dominant.

Note Since the cross section becomes so large in the forward region $\cos\theta \to 1$ due to the first term, the number of Bhabha scattering events can be used to measure the luminosity of the experiment.