Why is d^3p/E boost invariant?

Here $E^2 = p_x^2 + p_y^2 + p_z^2 + m^2$. Let's consider boosts in the x direction. Then p_y and p_z are invariant, so $dp_y\, dp_z$ is invariant. The question is then just why is dp_x/E boost invariant? Let's therefore separate off the transverse momentum components, and write this as

$$E^2 - p_x^2 = p_T^2 + m^2 =: E_T^2, \quad \text{where} \quad p_T^2 = p_y^2 + p_z^2. \tag{1}$$

The quantity p_T is called the *transverse momentum* squared. A general 4-momentum can then be written as a Lorentz boost of the one with $p_x = 0$:

$$(E, p_x, p_y, p_z) = (E_T \cosh\theta, E_T \sinh\theta, p_y, p_z). \tag{2}$$

Having used the hyperbolic angle θ to parametrize the possible values of p_x, we have $dp_x = E_T(\cosh\theta) d\theta$, and so

$$\frac{dp_x}{E} = d\theta. \tag{3}$$

Now observe that $d\theta$ is invariant under Lorentz boosts, just as $d\theta$ is invariant under rotations in the Euclidean plane. The way we might "prove" this in the Euclidean plane is to note that given two unit vectors $\vec{v}$ and $\vec{w}$, we have $\vec{v} \cdot \vec{w} = \cos\theta$. The left hand side is manifestly rotation invariant, so θ, the angle between the two vectors, is also rotation invariant. Then for infinitesimal angular separations, this says that $d\theta$ is invariant. Similarly for Lorentz transformations, with $\cos\theta$ replaced by $\cosh\theta$.