**4.3 Linear sigma model.** The interactions of pions at low energy can be described by a phenomenological model called the *linear sigma model.* Essentially, this model consists of $N$ real scalar fields coupled by a $\phi^4$ interaction that is symmetric under rotations of the $N$ fields. More specifically, let $\Phi^i(x)$, $i = 1, \ldots, N$ be a set of $N$ fields, governed by the Hamiltonian

$$H = \int d^3x \left( \tfrac{1}{2}(\Pi^i)^2 + \tfrac{1}{2}(\nabla\Phi^i)^2 + V(\Phi^2) \right),$$

where $(\Phi^i)^2 = \mathbf{\Phi} \cdot \mathbf{\Phi}$, and

$$V(\Phi^2) = \tfrac{1}{2}m^2(\Phi^i)^2 + \tfrac{\lambda}{4}\left((\Phi^i)^2\right)^2$$

is a function symmetric under rotations of $\mathbf{\Phi}$. For (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to $H$; hence, $V$ is the field potential energy.

(What does this Hamiltonian have to do with the strong interactions? There are two types of light quarks, $u$ and $d$. These quarks have identical strong interactions, but different masses. If these quarks are massless, the Hamiltonian of the strong interactions is invariant to unitary transformations of the 2-component object $(u, d)$:

$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow \exp(i\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}/2) \begin{pmatrix} u \\ d \end{pmatrix}.$$

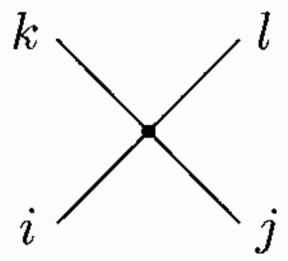
This transformation is called an *isospin* rotation. If, in addition, the strong interactions are described by a vector "gluon" field (as is true in QCD), the strong interaction Hamiltonian is invariant to the isospin rotations done separately on the left-handed and right-handed components of the quark fields. Thus, the complete symmetry of QCD with two massless quarks is $SU(2) \times SU(2)$. It happens that $SO(4)$, the group of rotations in 4 dimensions, is isomorphic to $SU(2) \times SU(2)$, so for $N = 4$, the linear sigma model has the same symmetry group as the strong interactions.)

From *An Introduction to Quantum Field Theory*, Peskin and Schroeder

**(a)** Analyze the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given above is exactly $N$ copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes as perturbation series in the parameter $\lambda$. Show that the propagator is

$$\overbrace{\Phi^i(x)\,\Phi^j}(y) = \delta^{ij}\, D_F(x-y),$$

where $D_F$ is the standard Klein-Gordon propagator for mass $m$, and that there is one type of vertex given by




$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}).$$

(That is, the vertex between two $\Phi^1$s and two $\Phi^2$s has the value $(-2i\lambda)$; that between four $\Phi^1$s has the value $(-6i\lambda)$.) Compute, to leading order in $\lambda$, the differential cross sections $d\sigma/d\Omega$, in the center-of-mass frame, for the scattering processes

$$\Phi^1\Phi^2 \to \Phi^1\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^2\Phi^2, \qquad \text{and} \qquad \Phi^1\Phi^1 \to \Phi^1\Phi^1$$

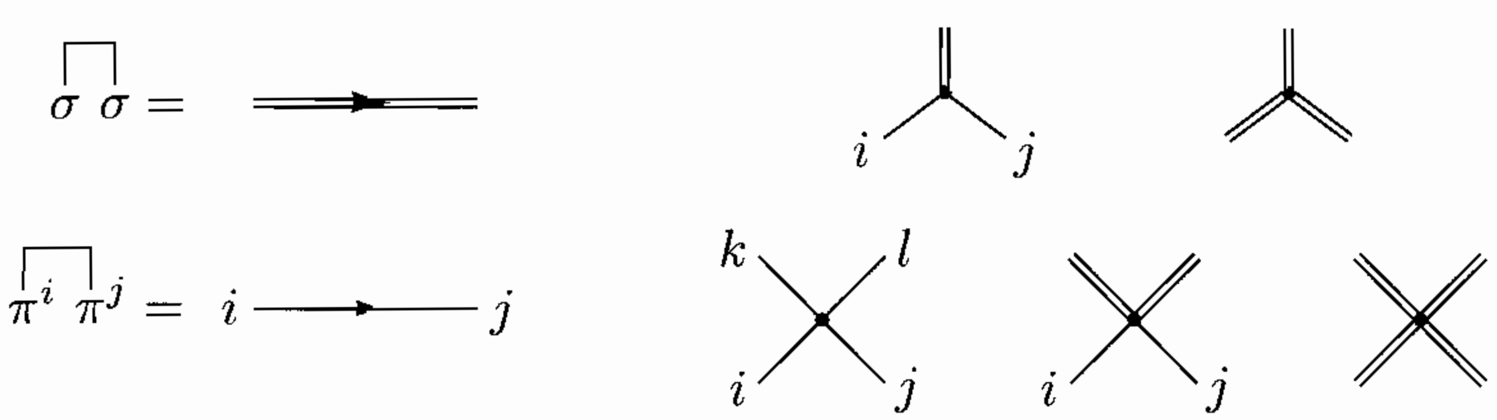
as functions of the center-of-mass energy.

Ans: The first two are lambda²/(16π²s), and the third is 9 times this.

**(b)** Now consider the case $m^2 < 0$: $m^2 = -\mu^2$. In this case, $V$ has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since $V$ is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting $\Phi^i$ toward the minimum of $V$. By rotational invariance, we can consider this shift to be in the $N$th direction. Write, then,

$$\Phi^i(x) = \pi^i(x), \quad i = 1, \ldots, N-1,$$
$$\Phi^N(x) = v + \sigma(x),$$

where $v$ is a constant chosen to minimize $V$. (The notation $\pi^i$ suggests a pion field and should not be confused with a canonical momentum.) Show that, in these new coordinates (and substituting for $v$ its expression in terms of $\lambda$ and $\mu$), we have a theory of a massive $\sigma$ field and $N-1$ *massless* pion fields, interacting through cubic and quartic potential energy terms which all become small as $\lambda \to 0$. Construct the Feynman rules by assigning values to the propagators and vertices:



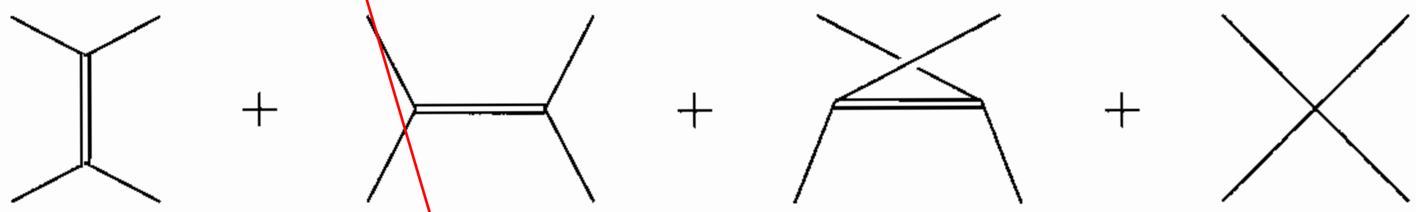

**(c)** Compute the scattering amplitude for the process

$$\pi^i(p_1)\,\pi^j(p_2) \;\to\; \pi^k(p_3)\,\pi^l(p_4)$$

This "at threshold" means (because the pions are massless) that one of the 4-momenta is zero. The reason for the vanishing of the amplitude when a pion momentum vanishes is that the pions defined here, to first order, represent the symmetry variations of the field Phi, i.e. the variations that don't change the potential, so there is an underlying "shift symmetry", but it is obscured by the way Phi is expanded. See section 28.2.1 of Schwartz to see, for the case N = 2, how one could define the fields to make the shift symmetry manifest. Even though the symmetry is not manifest here, it remains true that a pion does not interact with anything else unless its derivative - i.e. its momentum - is nonzero.



to leading order in $\lambda$. There are now four Feynman diagrams that contribute:



Show that, at threshold ($\mathbf{p}_i = 0$), these diagrams sum to *zero*. (Hint: It may be easiest to first consider the specific process $\pi^1\pi^1 \to \pi^2\pi^2$, for which only the first and fourth diagrams are nonzero, before tackling the general case.) Show that, in the special case $N = 2$ (1 species of pion), the term of $\mathcal{O}(p^2)$ also cancels.

**(d)** Add to $V$ a symmetry-breaking term,

$$\Delta V = -a\Phi^N,$$

where $a$ is a (small) constant. (In QCD, a term of this form is produced if the $u$ and $d$ quarks have the same nonvanishing mass.) Find the new value of $v$ that minimizes $V$, and work out the content of the theory about that point. Show that the pion acquires a mass such that $m_\pi^2 \sim a$, and show that the pion scattering amplitude at threshold is now nonvanishing and also proportional to $a$.