Problems from Timo Weigand

Problem 6.2 (Proof of Wick's Theorem):
In this exercise we prove Wick's theorem,

$$
\begin{aligned}
T\phi(x_1)\phi(x_2)\cdots\phi(x_n) = {} & :\phi(x_1)\phi(x_2)\cdots\phi(x_n): + \sum_{i<j} \overbracket{\phi(x_i)\phi}(x_j) :\phi(x_1)\cdots \\
& \cdots\phi(x_{i-1})\phi(x_{i+1})\cdots\phi(x_{j-1})\phi(x_{j+1})\cdots\phi(x_n): + \sum_{i<j,k<l,\dots} \overbracket{\phi(x_i)\phi}(x_j)\overbracket{\phi(x_k)\phi}(x_l) :\dots: + \\
& + \sum_{i<j,k<l,m<o,\dots} \overbracket{\phi(x_i)\phi}(x_j)\cdots\overbracket{\phi(x_m)\phi}(x_o) :\dots: + \dots
\end{aligned}
\tag{2}
$$

by mathematical induction, where the fields $\phi(x)$ are in the interaction picture. Therefore, they can be decomposed as $\phi(x) = \phi^+(x) + \phi^-(x)$ with

$$
\phi^+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}} e^{-ip\cdot x} \qquad \text{and} \qquad \phi^-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}} a^\dagger_{\mathbf{p}} e^{+ip\cdot x}\,. \tag{3}
$$

a.) Show that the contraction, which is defined as

$$
\overbracket{\phi(x)\phi}(y) := \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for } x^0 > y^0, \\ [\phi^+(y), \phi^-(x)] & \text{for } y^0 > x^0, \end{cases} \tag{4}
$$

is exactly the free Feynman propagator,

$$
\overbracket{\phi(x)\phi}(y) = D_F^{(0)}(x-y)\,. \tag{5}
$$

b.) **Base case**: Verify by explicitly evaluating the time-ordered product $T\phi(x)\phi(y)$ for $x^0 > y^0$ and $y^0 > x^0$ that

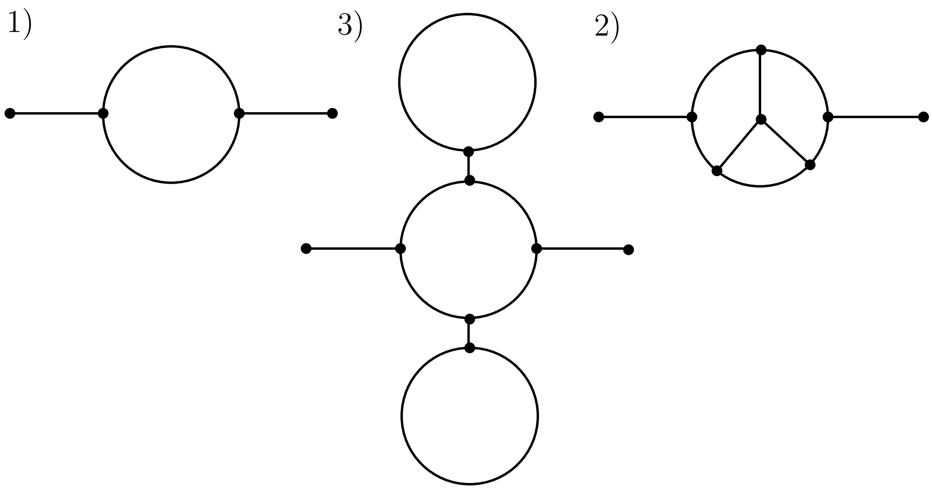
$$
T\phi(x)\phi(y) = \,:\phi(x)\phi(y): + \overbracket{\phi(x)\phi}(y). \tag{6}
$$

c.) **Inductive step**: Assume that (2) is true for m fields and calculate the time-ordered product for $m+1$ fields, $T\phi(x_1)\phi(x_2)\cdots\phi(x_{m+1})$. Without loss of generality, take $x_1^0 \geq \max(x_2^0, \ldots, x_{m+1}^0)$ because one can always relabel the space-time points such that this is the case. Now you may rewrite $T\phi(x_1)\phi(x_2)\cdots\phi(x_{m+1})$ as $\phi(x_1)T\phi(x_2)\cdots\phi(x_{m+1})$, use the assumption for m fields and bring the so obtained expression to a normal ordered form. The expression you get should agree with (2) for $n = m+1$ and, therefore, together with b.), proves Wick's theorem.

Problem 6.3 (Feynman rules in ϕ^3-theory):

a.) Give the Feynman rules for the propagator, the vertex and the external points in position-space and derive from these the Feynman rules in momentum-space for the $\lambda\,\phi^3$ theory, i.e. $\mathcal{L}_{\text{int}} = -\frac{\lambda}{3!}\phi^3$.

b.) Calculate the symmetry factors for the following diagrams:



Problem from M.D. Schwartz, QFT and the Standard Model

For H_0 it should read 1/2 [Pi^2 + (grad phi)^2].

7.4 We made a distinction between kinetic terms, which are bilinear in fields, and interactions, which have three or more fields. Time evolution with the kinetic terms is solved exactly as part of the free Hamiltonian. Suppose, instead, we only put the derivative terms in the free Hamiltonian and treated the mass as an interaction. So,

$$H_0 = \frac{1}{2}\phi\Box\phi, \quad H_{\text{int}} = \frac{1}{2}m^2\phi^2. \tag{7.125}$$

(a) Draw the (somewhat degenerate looking) Feynman graphs that contribute to the 2-point function $\langle 0|T\{\phi(x)\phi(y)\}|0\rangle$ using only this interaction, up to order m^6.

(b) Evaluate the graphs.

(c) Sum the series to all orders in m^2 and show you reproduce the propagator that would have come from taking $H_0 = \frac{1}{2}\phi\Box\phi + \frac{1}{2}m^2\phi^2$.