ASSIGNMENT 6

Due: 27.11.2013.

Problem 6.1 (LSZ reduction formula):
Revisit the derivation of the LSZ reduction formula as given in the lecture and work out in full
detail the transition
\[ \langle p_1, ..., p_n, \text{out} | \phi(x_1) | q_2, ..., q_r, \text{in} \rangle = \]
\[ i Z^{-1/2} \int d^4 y_1 e^{ip_1 \cdot y_1} (\square y_1 + m^2) \langle p_2, ..., p_n, \text{out} | T \phi(y_1) \phi(x_1) | q_2, ..., q_r, \text{in} \rangle \]
+ disconnected term.

Problem 6.2 (Proof of Wick’s Theorem):
In this exercise we prove Wick’s theorem,
\[ T \phi(x_1) \phi(x_2) \cdots \phi(x_n) = : \phi(x_1) \phi(x_2) \cdots \phi(x_n) : + \sum_{i < j} \phi(x_i) \phi(x_j) : \phi(x_1) \cdots \]
\[ \cdots \phi(x_{i-1}) \phi(x_{i+1}) \cdots \phi(x_{j-1}) \phi(x_{j+1}) \cdots \phi(x_n) : + \sum_{i < j, k < l, m < o, ...} \phi(x_i) \phi(x_j) \phi(x_k) \phi(x_l) \cdots : + \cdots \]
by mathematical induction, where the fields \( \phi(x) \) are in the interaction picture. Therefore, they
can be decomposed as \( \phi(x) = \phi^+(x) + \phi^-(x) \) with
\[ \phi^+(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ip \cdot x} \quad \text{and} \quad \phi^-(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^\dagger e^{ip \cdot x}. \]
a.) Show that the contraction, which is defined as
\[ \phi(x) \phi(y) := \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for } x^0 > y^0, \\ [\phi^+(y), \phi^-(x)] & \text{for } y^0 > x^0, \end{cases} \]
is exactly the free Feynman propagator,
\[ \phi(x) \phi(y) = D_F^{(0)}(x - y). \]
b.) Base case: Verify by explicitly evaluating the time-ordered product \( T \phi(x) \phi(y) \) for \( x^0 > y^0 \)
and \( y^0 > x^0 \) that
\[ T \phi(x) \phi(y) = : \phi(x) \phi(y) : + \phi(x) \phi(y). \]
c.) **Inductive step:** Assume that (2) is true for \( m \) fields and calculate the time-ordered product for \( m + 1 \) fields, \( T\phi(x_1)\phi(x_2)\cdots\phi(x_{m+1}) \). Without loss of generality, take \( x_1^0 \geq \max(x_2^0,\ldots,x_{m+1}^0) \) because one can always relabel the space-time points such that this is the case. Now you may rewrite \( T\phi(x_1)\phi(x_2)\cdots\phi(x_{m+1}) \) as \( \phi(x_1)T\phi(x_2)\cdots\phi(x_{m+1}) \), use the assumption for \( m \) fields and bring the so obtained expression to a normal ordered form. The expression you get should agree with (2) for \( n = m + 1 \) and, therefore, together with (2), proves Wick’s theorem.

**Problem 6.3 (Feynman rules in \( \phi^3 \)-theory):**

a.) Give the Feynman rules for the propagator, the vertex and the external points in position-space and derive from these the Feynman rules in momentum-space for the \( \lambda \phi^3 \) theory, i.e. \( \mathcal{L}_{\text{int}} = -\frac{\lambda}{3!}\phi^3 \).

b.) Calculate the symmetry factors for the following diagrams:

![Diagrams](image)

**Problem 6.4 (A 1-loop diagram in \( \phi^4 \)-theory):**

Consider the following diagram in \( \lambda \phi^4 \) theory (with \( \mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4 \)):

![Diagram](image)

Write down the expression for the corresponding Green’s function using Feynman rules in position space. Apply a Fourier transformation with respect to the coordinates of the four external points (\( x_i^\mu \rightarrow p_i^\mu \)) and show that the expression takes the form

\[
N(-i\lambda)^2(2\pi)^4\delta^{(4)}(p_4 + p_3 - p_2 - p_1) \prod_{i=1}^{4} \left( \frac{i}{p_i^2 - m_0^2 + i\epsilon} \right) \times \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_0^2 + i\epsilon} \left( \frac{i}{p - p_1 - p_2} \right)^2 \frac{i}{-m_0^2 + i\epsilon}.
\]

(7)

Note that the routing of momenta inside the loop is not unique, so your expression can differ from (7) up to shifts of the loop integration variable \( p \) by a linear combination of the external momenta. What is the value of \( N \)? Check that this expression is equal to the one obtained by directly applying Feynman rules in momentum space.
(a) Draw the three tree-level $e^-e^- \rightarrow e^-e^-$ diagrams following from this Lagrangian.

(b) Which one of the diagrams would be forbidden in real QED?

(c) Evaluate the other two diagrams, and express the answers in terms of $s$, $t$ and $u$.
   Give the diagrams an extra relative minus sign, because electrons are fermions.

(d) Now let us put back the spin. In the non-relativistic limit, the electron spin is conserved. This should be true at each vertex, since the photon is too soft to carry off any spin angular momentum. Thus, a vertex can only allow for $|\uparrow\rangle \rightarrow |\uparrow;\gamma\rangle$ or $|\downarrow\rangle \rightarrow |\downarrow;\gamma\rangle$. This forbids, for example, $|\uparrow\downarrow\rangle \rightarrow |\uparrow\uparrow\rangle$ from occurring. For each of the 16 possible sets of spins for the four electrons, we can show which processes are forbidden and which get contributions from the $s$-, $t$- or $u$-channels.

(e) It is difficult to measure electron spins. Thus, assume the beams are unpolarized, meaning that they have an equal fraction of spin-up and spin-down electrons, and that you do not measure the final electron spins, only the scattering angle $\theta$. What is the total rate $\frac{d\sigma}{d\cos \theta}$ you would measure? Express the answer in terms of $E_{CM}$ and $\theta$. Sketch the angular distribution.

7.4 We made a distinction between kinetic terms, which are bilinear in fields, and interactions, which have three or more fields. Time evolution with the kinetic terms is solved exactly as part of the free Hamiltonian. Suppose, instead, we only put the derivative terms in the free Hamiltonian and treated the mass as an interaction. So,

$$H_0 = \frac{1}{2} \phi \Box \phi, \quad H_{int} = \frac{1}{2} m^2 \phi^2. \quad (7.125)$$

(a) Draw the (somewhat degenerate looking) Feynman graphs that contribute to the 2-point function $\langle 0|T\{\phi(x)\phi(y)\}|0\rangle$ using only this interaction, up to order $m^6$.

(b) Evaluate the graphs.

(c) Sum the series to all orders in $m^2$ and show you reproduce the propagator that would have come from taking $H_0 = \frac{1}{2} \phi \Box \phi + \frac{1}{2} m^2 \phi^2$.

(d) Repeat the exercise classically: Solve for the massless propagator using an external current, perturb with the mass, sum the series, and show that you get the same answer as if you included the mass to begin with.

7.5 Show in general that integrating by parts does not affect matrix elements.

7.6 Use the Lagrangian

$$\mathcal{L} = -\frac{1}{2} \phi_1 \Box \phi_1 - \frac{1}{2} \phi_2 \Box \phi_2 + \frac{\lambda}{2} \phi_1 (\partial_\mu \phi_2)(\partial^\mu \phi_2) + \frac{g}{2} \phi_1^2 \phi_2 \quad (7.126)$$

to calculate the differential cross section

$$\frac{d\sigma}{d\Omega} (\phi_1 \phi_2 \rightarrow \phi_1 \phi_2) \quad (7.127)$$

at tree level.

7.7 Consider a Feynman diagram that looks like a regular tetrahedron, with the external lines coming out of the four corners. This can contribute to $2 \rightarrow 2$ scattering in a scalar field theory with interaction $\frac{\lambda}{4!} \phi^4$. You can take $\phi$ to be massless.