HW#3 —Phys624—Fall 2016 Prof. Ted Jacobson
due at beginning of class, Thursday 09/22/16 Room PSC 3151, (301)405-6020
www.physics.umd.edu/grt/taj/624b/ jacobson@umd.edu

1. Consider a quantum particle in one dimension, with Lagrangian

$$L = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}m_0^2 x^2 - \tfrac{1}{4!}gx^4. \tag{1}$$

The point of this problem is to explore the analogy with aspects of quantum field theory. Let $|0\rangle$ be the ground state of the "free" theory, i.e. with $g = 0$, and let $|\Omega\rangle$ be the ground state of the "interacting" theory, with $g \neq 0$. The first excited state in the free theory has energy m_0 relative to the ground state, whereas in the interacting theory the first excited state has, in general, some other energy m relative to the interacting ground state.

   (a) Evaluate the "Feynman propagator" $\langle 0|Tx(t)x(t')|0\rangle$ and its Fourier transform in the free theory, where $x(t)$ is the Heisenberg position operator.

   (b) Derive the analog of the spectral representation of $\langle \Omega|Tx(t)x(t')|\Omega\rangle$, now for the interacting theory, following the steps in Weigand's notes leading to (2.27).

   (c) Translate problem 2 in this homework into the equivalent problem for this 1d quantum particle, and solve all parts. To translate part (c), replace the preamble with "The spectral function can be written in the following form, where $\tilde{\rho}(M^2)$ consists of the contributions from all the higher energy states:"

   (d) Now change the interaction term from $\frac{1}{4!}gx^4$ to $\frac{1}{2}bx^2$, which is simple enough that you can solve the interacting theory exactly. Show explicitly that for this theory $Z = 1$, and explain how this is consistent with the general result that $Z = 1$ if and only if the theory is free.

2. **Proof that $Z < 1$ unless the theory is free** - see attached.

2. Proof that Z < 1 unless the theory is free

Problem from Timo Weigand

From the lecture we know that

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\,\rho(M^2)\,D_F(x-y,M^2)\,. \tag{12}$$

a.) Show with the definitions

$$D(x-y) := \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-i\,p(x-y)} \text{ and } D_F(x-y) := \theta(x_0-y_0)D(x-y)+\theta(y_0-x_0)D(y-x)$$

that

$$\lim_{x_0-y_0\to 0_+} \partial_{x_0} D_F(x-y) = -\frac{i}{2}\delta^3(\mathbf{x}-\mathbf{y}) = -\lim_{y_0-x_0\to 0_+} \partial_{x_0} D_F(x-y).$$

b.) With the canonical commutation relations

$$[\phi(t,\mathbf{x}),\dot{\phi}(t,\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}), \tag{13}$$

equation (12) and point a.), show that

$$\int_0^\infty \frac{dM^2}{2\pi}\,\rho(M^2) = 1. \tag{14}$$

Hint: Express the commutator in terms of a pair of limits of time-ordered products

c.) For a theory with just one kind of one-particle state and no bound states, the spectral function takes the following form:

$$\rho(M^2) = (2\pi\,Z)\delta(M^2-m^2)+\tilde{\rho}(M^2)\,. \tag{15}$$

Show with (15) and (14) that $Z \leq 1$ and that $Z = 1$ if and only if $\tilde{\rho}(M^2) = 0$.

d.) Show that $\tilde{\rho}(M^2) = 0$ implies $(\partial^2+m^2)\phi(x)|\Omega\rangle = 0$. Now, the Reeh-Schlieder theorem for a *local* operator $\mathcal{O}$ states that $\mathcal{O}|\Omega\rangle = 0$ implies $\mathcal{O} = 0$.[2] Thus, $Z = 1$ if and only if the field $\phi(x)$ is a free field.

[2] This is not in contradiction with $P_\mu|\Omega\rangle = 0$ because P_μ involves integrals.