HW#12 —Phys624—Fall 2016 Prof. Ted Jacobson
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This is Problem 4 only. Three other problems are listed at the course webpage.

Why a massless spin-1 field with Lorentz symmetry must have gauge symmetry

a) Show that a transformation $x^\alpha \to \Lambda^\alpha{}_\mu x^\mu$ preserves the invariant interval $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ if and only if

$$\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} = \eta_{\mu\nu}. \tag{1}$$

These are *Lorentz transformations.*

b) Show that if $\Lambda^\alpha{}_\mu = \delta^a{}_\mu + \lambda^\alpha{}_\mu$ is to be a Lorentz transformation with $\lambda^\alpha{}_\mu$ infinitesimal, then

$$\lambda^\alpha{}_\mu \eta_{\alpha\nu} + \lambda^\alpha{}_\nu \eta_{\mu\alpha} = 0. \tag{2}$$

If we adopt the convention that indices are raised or lowered by contraction with the Minkowski metric, this is equivalent to antisymmetry of $\lambda_{\mu\nu}$,

$$\lambda_{\mu\nu} + \lambda_{\nu\mu} = 0. \tag{3}$$

$\lambda^\mu{}_\nu$ is called an *infinitesimal generator* of a Lorentz transformation, or *Lorentz generator* for short. For any pair of vectors A^μ and B^μ, the tensor $A^\mu B_\nu - B^\mu A_\nu$ is a Lorentz generator, and the collection of six such generators constructed using a basis of 4-vectors span the linear space of Lorentz generators.

c) Find the Lorentz generators of the *little group* of a null 4-momentum vector $P^\mu = (1, 0, 0, 1)$, using the basis consisting of P^μ, $T^\mu = (1, 0, 0, 0)$, $X^\mu = (0, 1, 0, 0)$ and $Y^\mu = (0, 0, 1, 0)$. Show that one of these generates a rotation about the z axis and the other two commute with each other. The latter two generate *null rotations*: they consist of a boost in one direction, combined with a rotation through the same angle in a plane containing that direction. The null rotations form a noncompact (unbounded) group, so if they are nontrivially represented as unitary transformations in the Hilbert space, then the space of polarizations must be infinite dimensional. This is impossible for a vector field which has a polarization 4-vector. (Moreover, no such particles are observed in nature, but see http://arxiv.org/abs/1404.0675 for recent progress defining a field theory of such objects.) So, for a vector field, the null rotations must somehow be trivially represented in Hilbert space.

d) Show that a polarization vector ϵ^μ is invariant under the null rotations only if $\epsilon \propto P$, which is invariant under the *entire* little group. (This would describe a particle with no internal spin, i.e. a scalar, and would imply that the vector field A_μ is the gradient of a scalar field, $A_\mu = \partial_\mu \lambda$, reducing the vector to (at most) a scalar degree of freedom.)

e) Show that if $P \cdot \epsilon = 0$, then $\delta\epsilon \sim P$ under null rotations. This change can be identified with a gauge transformation, $\delta A_\mu = \partial_\mu \lambda$, and thereby declared unphysical.