carefully
spin up. Has its helicity
consistent with part (a)?

(e) How can you measure the spin of a slow electron?

(f) Suppose you have a radioactive source, such as cobalt-60, which undergoes β-decay ${}^{60}_{27}\text{Co} \to {}^{60}_{28}\text{Ni} + e^- + \bar{\nu}$. How could you (in principle) find out if those electrons coming out are polarized; that is, if they all have the same helicity? Do you think they would be polarized? If so, which polarization do you expect more of?

11.7 Show that the most general Lagrangian term you can write down in terms of Dirac spinors, γ-matrices, and the photon field A_μ is automatically invariant under CPT. To warm up, consider first the terms in Eq. (11.91).

11.8 Fierz rearrangement formulas (Fierz identities). It is often useful to rewrite spinor contractions in other forms to simplify formulas. Show that

(a) $(\bar{\psi}_1\gamma^\mu P_L\psi_2)(\bar{\psi}_3\gamma^\mu P_L\psi_4) = -(\bar{\psi}_1\gamma^\mu P_L\psi_4)(\bar{\psi}_3\gamma^\mu P_L\psi_2)$

(b) $(\bar{\psi}_1\gamma^\mu\gamma^\alpha\gamma^\beta P_L\psi_2)(\bar{\psi}_3\gamma^\mu\gamma^\alpha\gamma^\beta P_L\psi_4) = -16(\bar{\psi}_1\gamma^\mu P_L\psi_4)(\bar{\psi}_3\gamma^\mu P_L\psi_2)$

(c) $\text{Tr}\left[\Gamma^M\Gamma^N\right] = 4\delta^{MN}$, with $\Gamma^M \in \{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma_5\gamma^\mu, \gamma_5\}$

(d) $(\bar{\psi}_1\Gamma^M\psi_2)(\bar{\psi}_3\Gamma^N\psi_4) = \sum_{PQ}\frac{1}{16}\text{Tr}\left[\Gamma^P\Gamma^M\Gamma^Q\Gamma^N\right](\bar{\psi}_1\Gamma^P\psi_4)(\bar{\psi}_3\Gamma^Q\psi_2)$

where $P_L = \frac{1-\gamma_5}{2}$ projects out the left-handed spinor from a Dirac fermion. The identities with P_L play an important role in the theory of weak interactions, which only involves left-handed spinors (see Chapter 29).

11.9 The electron neutrino is a nearly massless neutral particle. Its interactions violate parity: only the left-handed neutrino couples to the W and Z bosons. No one has ever seen a right-handed neutrino, and we do not (yet) know if they exist. Neutrino masses were conclusively discovered through oscillation experiments in the 1990's.

(a) One way to give neutrinos mass is to imagine that there exist right-handed neutrinos which have no interactions. Such particles are called **sterile neutrinos**. Then one can write the kinetic Lagrangian for the neutrinos as

11.10

$$\mathcal{L}_{\text{kin}} = i\nu_L^\dagger \bar{\sigma}^\mu \partial_\mu \nu_L + i\nu_R^\dagger \sigma^\mu \partial_\mu \nu_R - m(\nu_L^\dagger \nu_R + \nu_R^\dagger \nu_L) + i\frac{M}{2}\left(\nu_R^T \sigma_2 \nu_R - \nu_R^\dagger \sigma_2 \nu_R^\star\right), \tag{11.93}$$

Here, ν_L is a left-handed $(\frac{1}{2}, 0)$ two-component Weyl spinor and ν_R is a right-handed $(0, \frac{1}{2})$ Weyl spinor. Note that there are two mass terms: a Dirac mass m, as for the electron, and a Majorana mass, M. Show that ~~this Lagrangian is Lorentz invariant and that~~ $\chi_L \equiv i\sigma_2 \nu_R^\star$ transforms as a left-handed spinor under the Lorentz group, so that it can mix with ν_L.

(b) What are the mass eigenstates? That is, find linear combinations ψ_1 and ψ_2 of χ_L and ν_L that satisfy the Klein-Gordon equation $(\Box + m_i^2)\psi_i = 0$. What are m_i?

For (b): It seems convenient to combine the two left handed fermions into a doublet, and write the first order field equation in terms of the doublet, with a mass matrix term. Then find the eigenvalues and eigenvectors of the matrix, and simplify them using M >> m. Then show that the mass eigenstates satisfy the Klein-Gordon equation. (For this step use the hint written out on the hw web page.)

(c) Suppose $M \gg m$. For example, $M = 10^{10}$ GeV and $m = 100$ GeV. What are the masses of the physical particles? The fact that as M goes up, the physical masses go down, inspired the name **see-saw mechanism** for this neutrino mass arrangement. What other choice of M and m would give the same spectrum of observed particles (i.e. particles less than ~1 TeV)?

(c): Change this part to: If m=100GeV, what value of M would yield a light neutrino mass 0.1 eV, and what would be the heavy neutrino mass?

(d) The left-handed neutrino couples to the Z boson and also to the electron through the W boson. The W boson also couples the neutron and proton. The relevant part for the weak-force Lagrangian is

$$\mathcal{L}_{\text{weak}} = g_W(\nu_L^\dagger \not{W} e_L + e_L^\dagger \not{W} \nu_L) + g_Z(\nu_L^\dagger \not{Z} \nu_L) + g_W(n \not{W} \bar{p} + \bar{n} \not{W} p). \tag{11.94}$$

Using these interactions, draw a Feynman diagram for neutrinoless double β-decay, in which two neutrons decay to two protons and two electrons.

(e) Which of the terms in $\mathcal{L}_{\text{kin}}$ and $\mathcal{L}_{\text{weak}}$ respect a global symmetry (lepton number) under which $\nu_L \to e^{i\theta}\nu_L$, $\nu_R \to e^{i\theta}\nu_R$ and $e_L \to e^{i\theta}e_L$? Define arrows on the e and ν lines to respect lepton number flow. Show that you cannot connect the arrows on your diagram without violating lepton number. Does this imply that neutrinoless double β-decay can tell if the neutrino has a Majorana mass?

10 In Section 10.4, we showed that the electron has a magnetic dipole moment, of order $\mu_B = \frac{e}{2m_e}$, by squaring the Dirac equation. An additional magnetic moment could come from an interaction of the form $\mathcal{B} = iF_{\mu\nu}\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi$ in the Lagrangian. An electric dipole moment (EDM) corresponds to a term of the form