4.3 Linear sigma model. The interactions of pions at low energy can be described by a phenomenological model called the *linear sigma model*. Essentially, this model consists of N real scalar fields coupled by a ϕ^4 interaction that is symmetric under rotations of the N fields. More specifically, let $\Phi^i(x),\ i = 1, \ldots, N$ be a set of N fields, governed by the Hamiltonian

$$H = \int d^3x \left(\tfrac{1}{2}(\Pi^i)^2 + \tfrac{1}{2}(\nabla\Phi^i)^2 + V(\Phi^2)\right),$$

where $(\Phi^i)^2 = \mathbf{\Phi}\cdot\mathbf{\Phi}$, and

$$V(\Phi^2) = \tfrac{1}{2}m^2(\Phi^i)^2 + \tfrac{\lambda}{4}\left((\Phi^i)^2\right)^2$$

is a function symmetric under rotations of $\mathbf{\Phi}$. For (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to H; hence, V is the field potential energy.

(What does this Hamiltonian have to do with the strong interactions? There are two types of light quarks, u and d. These quarks have identical strong interactions, but different masses. If these quarks are massless, the Hamiltonian of the strong interactions is invariant to unitary transformations of the 2-component object (u, d):

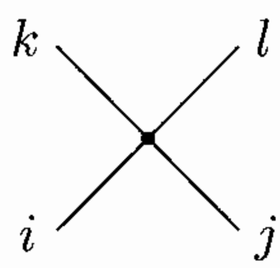
$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow \exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}/2)\begin{pmatrix} u \\ d \end{pmatrix}.$$

This transformation is called an *isospin* rotation. If, in addition, the strong interactions are described by a vector "gluon" field (as is true in QCD), the strong interaction Hamiltonian is invariant to the isospin rotations done separately on the left-handed and right-handed components of the quark fields. Thus, the complete symmetry of QCD with two massless quarks is $SU(2) \times SU(2)$. It happens that $SO(4)$, the group of rotations in 4 dimensions, is isomorphic to $SU(2) \times SU(2)$, so for $N = 4$, the linear sigma model has the same symmetry group as the strong interactions.)

(a) Analyze the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given above is exactly N copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes as perturbation series in the parameter λ. Show that the propagator is

$$\Phi^i(x)\,\Phi^j(y) = \delta^{ij}\, D_F(x-y),$$

where D_F is the standard Klein-Gordon propagator for mass m, and that there is one type of vertex given by



$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}).$$

(That is, the vertex between two Φ^1s and two Φ^2s has the value $(-2i\lambda)$; that between four Φ^1s has the value $(-6i\lambda)$.) Compute, to leading order in λ, the differential cross sections $d\sigma/d\Omega$, in the center-of-mass frame, for the scattering processes

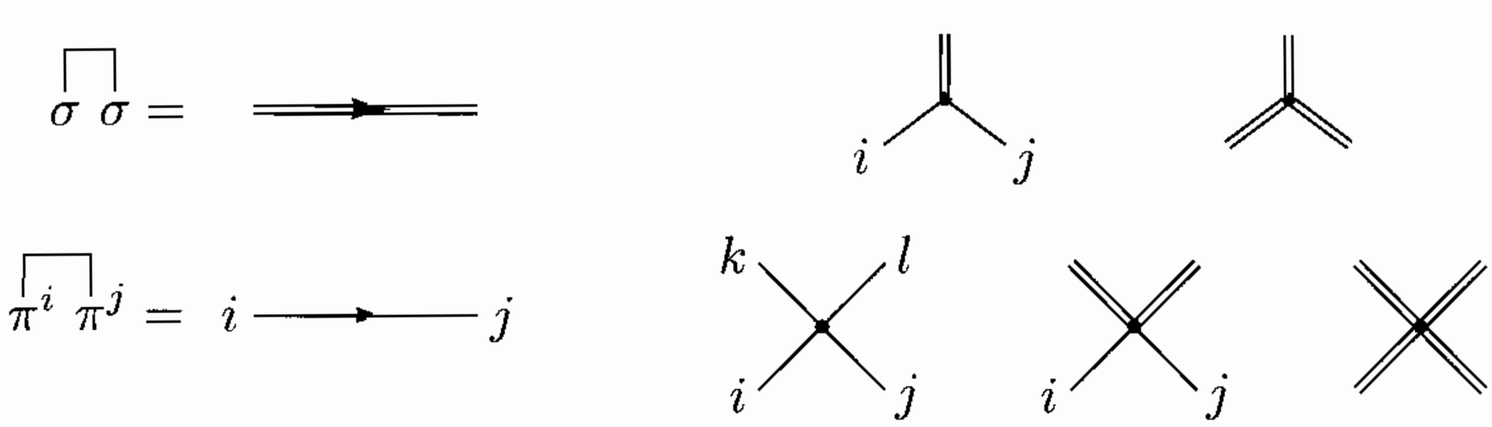
$$\Phi^1\Phi^2 \to \Phi^1\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^2\Phi^2, \qquad \text{and} \qquad \Phi^1\Phi^1 \to \Phi^1\Phi^1$$

as functions of the center-of-mass energy.

(b) Now consider the case $m^2 < 0$: $m^2 = -\mu^2$. In this case, V has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since V is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting Φ^i toward the minimum of V. By rotational invariance, we can consider this shift to be in the Nth direction. Write, then,

$$\Phi^i(x) = \pi^i(x), \quad i = 1, \ldots, N-1,$$
$$\Phi^N(x) = v + \sigma(x),$$

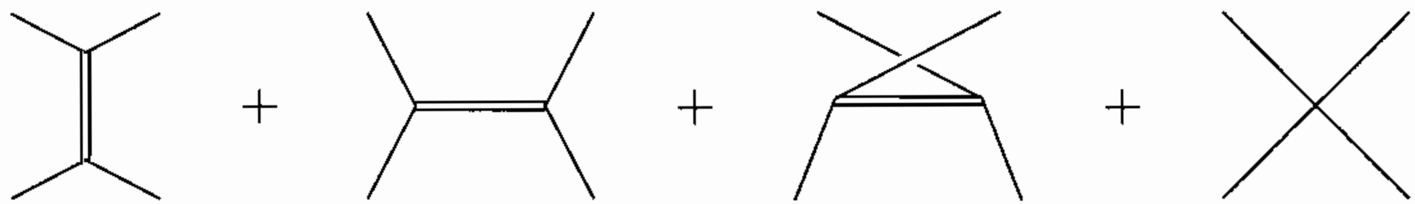
where v is a constant chosen to minimize V. (The notation π^i suggests a pion field and should not be confused with a canonical momentum.) Show that, in these new coordinates (and substituting for v its expression in terms of λ and μ), we have a theory of a massive σ field and $N-1$ *massless* pion fields, interacting through cubic and quartic potential energy terms which all become small as $\lambda \to 0$. Construct the Feynman rules by assigning values to the propagators and vertices:



(c) Compute the scattering amplitude for the process

$$\pi^i(p_1)\,\pi^j(p_2) \;\to\; \pi^k(p_3)\,\pi^l(p_4)$$

to leading order in λ. There are now four Feynman diagrams that contribute:



Show that, at threshold ($\mathbf{p}_i = 0$), these diagrams sum to *zero*. (Hint: It may be easiest to first consider the specific process $\pi^1\pi^1 \to \pi^2\pi^2$, for which only the first and fourth diagrams are nonzero, before tackling the general case.) Show that, in the special case $N = 2$ (1 species of pion), the term of $\mathcal{O}(p^2)$ also cancels.

(d) Add to V a symmetry-breaking term,

$$\Delta V = -a\Phi^N,$$

where a is a (small) constant. (In QCD, a term of this form is produced if the u and d quarks have the same nonvanishing mass.) Find the new value of v that minimizes V, and work out the content of the theory about that point. Show that the pion acquires a mass such that $m_\pi^2 \sim a$, and show that the pion scattering amplitude at threshold is now nonvanishing and also proportional to a.