HW#5 —Phys624—Fall 2015 Prof. Ted Jacobson
due at beginning of class, Thursday 010/15/15 Room PSC 3151, (301)405-6020
www.physics.umd.edu/grt/taj/624a/ jacobson@umd.edu

This is Problem 1 only. Two other problems are listed at the course webpage.

1. *Relation between gauge and Lorentz invariance for a massless spin-1 field*

We saw in class that the action for the massless limit of a massive spin-1 (Proca) field has gauge symmetry. In this problem, you'll see why the gauge symmetry is required.

a) Show that a transformation $x^\alpha \to \Lambda^\alpha{}_\mu x^\mu$ preserves the invariant interval $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ if and only if

$$\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} = \eta_{\mu\nu}. \tag{1}$$

These are *Lorentz transformations.*

b) Show that if $\Lambda^\alpha{}_\mu = \delta^a{}_\mu + \lambda^\alpha{}_\mu$ is to be a Lorentz transformation with $\lambda^\alpha{}_\mu$ infinitesimal, then

$$\lambda^\alpha{}_\mu \eta_{\alpha\nu} + \lambda^\alpha{}_\nu \eta_{\mu\alpha} = 0. \tag{2}$$

If we adopt the convention that indices are raised or lowered by contraction with the Minkowski metric, this is equivalent to antisymmetry of $\lambda_{\mu\nu}$,

$$\lambda_{\mu\nu} + \lambda_{\nu\mu} = 0. \tag{3}$$

$\lambda^\mu{}_\nu$ is called an *infinitesimal generator* of a Lorentz transformation, or *Lorentz generator* for short. For any pair of vectors A^μ and B^μ, the tensor $A^\mu B_\nu - B^\mu A_\nu$ is a Lorentz generator, and the collection of six such generators constructed using a basis of 4-vectors span the linear space of Lorentz generators.

c) Find the Lorentz generators of the *little group* of a null 4-momentum vector $P^\mu = (1, 0, 0, 1)$, using the basis consisting of P^μ, $T^\mu = (1, 0, 0, 0)$, $X^\mu = (0, 1, 0, 0)$ and $Y^\mu = (0, 0, 1, 0)$. Show that one of these generates a rotation about the z axis and the other two commute with each other. The latter two are called *null rotations*; they consist of a combined boost in one direction and a rotation in a plane containing that direction. It turns out that if the null rotations are nontrivially represented as unitary transformations in the Hilbert space, then the space of polarizations is infinite dimensional. This is impossible for a vector field which has a polarization 4-vector. (Moreover, no such particles are observed in nature, but see http://arxiv.org/abs/1404.0675 for recent progress defining a field theory of such objects.) So, for a vector field, the null rotations must somehow be trivially represented in Hilbert space.

d) If $\lambda^\mu{}_\nu$ is a Lorentz generator, then $(\exp\lambda)^\mu{}_\nu$ is a Lorentz transformation. Find the Lorentz transformations generated by the null rotations, and show that they form a non-compact group equivalent to the group of translations in a plane.

e) Show that a polarization vector ϵ^μ is invariant under the null rotations only if $\epsilon \propto P$, which is invariant under the *entire* little group. This would describe a particle with no internal spin, i.e. a scalar, and would imply that the vector field A_μ is the gradient of a scalar field, $A_\mu = \partial_\mu \lambda$, reducing the vector to (at most) a scalar degree of freedom.

f) Show that if $P \cdot \epsilon = 0$, then $\delta\epsilon \sim P$ under null rotations. This change can be identified with a gauge transformation, $\delta A_\mu = \partial_\mu \lambda$, and thereby declared unphysical.