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Irreducible tensor operators and the Wigner-Eckart theorem

Vector operators

The rotation $\mathcal{O} \to \mathcal{O}'$ of an observable is defined so that $\langle\psi|\mathcal{O}'|\psi\rangle = \langle\psi'|\mathcal{O}|\psi'\rangle$, where $|\psi'\rangle = \exp(-i\vec{\theta}\cdot\vec{J})|\psi\rangle$ is the rotated state. Thus

$$\mathcal{O}' = e^{i\theta^b J^b}\mathcal{O}e^{-i\theta^b J^b} = \mathcal{O} + i\theta^b[J^b, \mathcal{O}] + O(\theta^2), \tag{1}$$

where J^a are the generators of rotation (and are the angular momentum), and repeated indices are summed over. A *vector operator* V^a transforms under rotations as does a vector in Euclidean space,

$$V^a \to V^a + \epsilon^{abc}\theta^b V^c + O(\theta^2), \tag{2}$$

which is implemented by (1) provided

$$[J^a, V^b] = i\epsilon^{abc}V^c. \tag{3}$$

In the case where $V^b = J^b$, these are the commutation relations of the generators of the rotation group. This is consistent, of course, since J^b is indeed a vector. Other examples of vector operators are position, momentum, orbital angular momentum, electric dipole moment, spin, magnetic moment, etc. Note that a "vector operator" can be defined relative to rotations on a subspace of Hilbert space, so J^a in (3) need not be the total angular momentum.

The commutation relations (3) are equivalent to

$$[J_z, V_q] = q\,V_q \tag{4}$$

$$[J_\pm, V_q] = \sqrt{2}\,V_{q\pm1}, \tag{5}$$

where

$$V_0 = V^z \quad \text{and} \quad V_{\pm1} = \mp(V^x \pm iV^y)/\sqrt{2} \tag{6}$$

are the so-called "spherical components" of V^a, and $V_q = 0$ for $|q| > 1$.

An example is the position vector r^a. The Cartesian components of this vector are (x, y, z). The spherical components, in Cartesian and polar coordinates, are

$$r_0 = z = r\cos\theta, \quad \text{and} \quad r_{\pm1} = \mp(x \pm iy)/\sqrt{2} = \mp r\sin\theta e^{\pm i\phi}/\sqrt{2}, \tag{7}$$

which are directly related to the $l = 1$ spherical harmonics via

$$r_q = \sqrt{\tfrac{4\pi}{3}}\,rY_{1q}(\theta, \phi). \tag{8}$$

Tensor operators

An *(irreducible) tensor operator* of order $k = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ is a collection of operators T_{kq}, $q = k, k-1, \ldots, -k$, that transforms under rotations as

$$\boxed{\begin{aligned} [J_z, T_{kq}] &= q\, T_{kq} \\ [J_\pm, T_{kq}] &= \sqrt{k(k+1) - q(q\pm 1)}\, T_{k,q\pm 1} \end{aligned}} \qquad \begin{aligned} (9) \\ (10) \end{aligned}$$

with $T_{kq} \equiv 0$ when $|q| > k$. For $k = 1$ these coincide with the vector operator relations (4,5). For integer k, they apply to the spherical harmonics $Y_{kq}(\theta, \phi)$, considered as multiplication operators, A useful example is the operator of tensor multiplication by some spin-k multiplet of states $|kq\rangle$, i.e.

$$M_{kq}|\psi\rangle := |kq\rangle|\psi\rangle. \tag{11}$$

This is a bit odd, in that it maps the ket $|\psi\rangle$ into a larger Hilbert space, but that's ok. M_{kq} satisfies the commutation relations (9,10) simply because $J_z|kq\rangle = q|kq\rangle$ and $J_\pm|kq\rangle = \sqrt{k(k+1) - q(q\pm 1)}|k, q\pm 1\rangle$. *Mutiplication by a tensor operator adds angular momentum to the state on which it acts.*

The commutation relations (9,10) imply that the set of vectors $\{T_{kq}|jm\rangle\}$, for fixed k and j, is closed under the action of J_z and $J_\pm$, hence can be decomposed into a set of irreducible representations of the rotation group. In particular, (9) implies

$$J_z T_{kq}|jm\rangle = (q+m) T_{kq}|jm\rangle, \tag{12}$$

so the decomposition proceeds just as for the product space spanned by the vectors $\{|kq\rangle|jm\rangle\}$. That is, starting with the top state, $T_{kk}|jj\rangle$, successive application of J_- produces all the states in a complete spin-$(k+j)$ representation. Next, the orthogonal state with total J_z equal to $k+j-1$ is the top state of a spin-$(k+j-1)$ representation, and so on. This yields a sum of representations, $(k+j) \oplus (k+j-1) \oplus \cdots \oplus |k-j|$. It follows from this structure that we have the following *selection rules*:

$$\boxed{\langle\omega' j' m'|T_{kq}|\omega j m\rangle = 0 \quad \text{unless} \quad j' \subset k \otimes j \quad \text{and} \quad m' = q + m} \tag{13}$$

These rules are the single most important takeaway of this whole business.

Wigner-Eckart Theorem

Let J^2, J_z, and Ω form a complete commuting set of operators with corresponding eigenstates labeled uniquely by $|\omega j m\rangle$, where Ω stands for a collection of operators and ω for the corresponding eigenvalues. According to the selection rule (13), each (m', m) pair determines a unique value of q for which T_{kq} can have a nonzero matrix element. The commutation relations (10) imply linear relations among the (possibly) nonzero matrix elements. Omitting the $\omega' j'$ and ωj labels, which play no role here, these relations are

$$a\,\langle m'|T_{k,q\pm 1}|m\rangle = b\,\langle m' \mp 1|T_{kq}|m\rangle - c\,\langle m'|T_{kq}|m \pm 1\rangle, \tag{14}$$

with

$$\begin{aligned} a &= \sqrt{k(k+1) - q(q \pm 1)} \qquad (15)\\ b &= \sqrt{j'(j'+1) - m'(m' \mp 1)} \qquad (16)\\ c &= \sqrt{j(j+1) - m(m \pm 1)}. \qquad (17) \end{aligned}$$

The matrix elements $\langle \omega' j' m' | T_{kq} | \omega j m \rangle$ with fixed $\omega' j' \omega j$ are therefore determined recursively by (for example) the nonzero matrix element with maximal m' and m. (One need not work out the formula explicitly for each matrix element to see that the elements are so determined.) Thus, *the $m'm$ matrix elements of any two tensor operators of the same order k are proportional to each other*, in the sense that

$$\langle \omega_1' j' m' | T^{(1)}_{kq} | \omega_1 j m \rangle = S \, \langle \omega_2' j' m' | T^{(2)}_{kq} | \omega_2 j m \rangle, \qquad (18)$$

where S is a scalar that depends on $\omega_1', \omega_1, \omega_2', \omega_2, j', j$ and the operators $T^{(1)}$ and $T^{(2)}$, but not on m', m, q. In writing (18) we have assumed of course that the relevant matrix elements of $T^{(2)}_{kq}$ don't vanish identically.

The matrix elements of the tensor multiplication operator (11), are just the Clebsch-Gordan coefficients $\langle j' m' | kjqm \rangle$. Choosing $T^{(2)}_{kq} = M_{kq}$ in (18) thus shows in particular that, for any tensor operator T_{kq},

$$\boxed{\langle \omega' j' m' | T_{kq} | \omega j m \rangle = \langle \omega' j' || T_k || \omega j \rangle \, \langle j' m' | kjqm \rangle} \qquad (19)$$

where the coefficient $\langle \omega' j' || T || \omega j \rangle$ is called the "reduced matrix element".[1] This is the *Wigner-Eckart theorem.* It states that the matrix elements of an irreducible tensor operator are proportional to the Clebsch-Gordan (CG) coefficients, with a coefficient that depends on ω', ω, j', j, but not on m', m, q. The argument given so far only shows that (19) holds when the CG coefficients do not vanish, but this is sufficient, given the selection rule (13), which states that the matrix element vanishes whenever the CG coefficient does.

Vector Projection Theorem

As a special case of (18), the matrix elements of any vector operator V^a between states of the *same*[2] j are proportional to those of J^a:

$$\langle \omega' j m' | V^a | \omega j m'' \rangle = S_{\omega' \omega j} \, \langle \omega j m' | J^a | \omega j m'' \rangle. \qquad (20)$$

To evaluate $S_{\omega' \omega j}$, multiply (20) by $\langle \omega j m'' | J^a | \omega j m' \rangle$ and sum over m''. This yields

$$\langle \omega' j m' | V^a | \omega j m'' \rangle = \frac{\langle \omega' j m | \vec{V} \cdot \vec{J} | \omega j m \rangle}{j(j+1)} \, \langle \omega j m' | J^a | \omega j m'' \rangle, \qquad (21)$$

which is called the *projection theorem.* Thus, within a given j representation, only the component of $\vec{V}$ parallel to $\vec{J}$ contributes to the matrix elements of V^a.

[1] Sometimes $1/\sqrt{2j'+1}$ is factored out in the definition of the reduced matrix element.

[2] The restriction to matrix elements between states of the *same* j is in general necessary for (20) to be true, since the matrix elements of J^a between different j's vanish, but those of V^a do not in general.

Landé g-factor

A useful application of the projection theorem is to express the magnetic moment of a system in terms of the total angular momentum. Consider for example an atom with many electrons. The magnetic moment is

$$\vec{\mu} = -(\mu_B/\hbar)\sum_i(\vec{L}_i + g_s\vec{S}_i) = -(\mu_B/\hbar)[\vec{J} + (g_s - 1)\vec{S}], \tag{22}$$

where the sum is over the different electrons, g_s is the electron g-factor, and $\vec{J} = \vec{L} + \vec{S}$ is the total orbital plus spin angular momentum of the electrons. The magnetic moment is a vector operator with respect to the total electronic angular momentum $\vec{J}$, so the Wigner-Eckart theorem tells us that, within a fixed irreducible representation of this angular momentum, we have

$$\langle\omega' JM'|\vec{\mu}|\omega JM\rangle = -g_J\,\frac{\mu_B}{\hbar}\,\langle\omega' JM'|\vec{J}|\omega JM\rangle \tag{23}$$

for some coefficient g_J. The projection theorem applied to $\vec{\mu}$, with the help of $\vec{S}\cdot\vec{J} = [J^2 + S^2 - (J-S)^2]/2$, yields

$$g_J = 1 + (g_s - 1)\frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}. \tag{24}$$

This is called the *Landé g-factor*.

Trace Theorem

The trace of the matrix of T_{k0} $(k \neq 0)$ in a subspace of given ω and j is zero:

$$\sum_m \langle\omega jm|T_{k0}|\omega jm\rangle = 0. \tag{25}$$

This follows from (i) the commutation relation $[J_+, T_{k,-1}] \propto T_{k0}$ (10), which can be truncated to the given j subspace since J_+ acts within the subspace, and (ii) the fact that the trace of a commutator of finite dimensional matrices vanishes.

Hole-Particle equivalence

In some ways, a shell filled with identical fermions except for n "holes" behaves the same as a shell with only n such particles. More precisely, let $T_{k0}(i)$ be a single particle tensor operator with $k > 0$, indexed by the particle label i. It can be shown that

$$\langle j^{2j+1-n}JM|\sum_{i=n+1}^{2j+1} T_{k0}(i)|j^{2j+1-n}JM\rangle = (-1)^{k+1}\langle j^n JM|\sum_{i=1}^{n} T_{k0}(i)|j^n JM\rangle, \tag{26}$$

where $|j^n JM\rangle$ is a totally antisymmetric state of n identical fermions, each with angular momentum j, adding up to a total angular momentum J and total z-component of angular momentum M. (For a proof, see for example *Nuclear Shell Theory*, A. de Shalit and I. Talmi (Academic Press, 1963).)