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Nuclear multipole moments and symmetries

Since a nucleon (neutron or proton), or a compound nucleus, contains a charge distribution, it might have electric and/or magnetic multipole moments. These are strongly constrained by symmetry considerations. For example, while a neutron has a magnetic dipole moment, and might have an electric dipole moment, it cannot have a magnetic quadrupole moment. This follows from the selection rule aspect of the Wigner-Eckart theorem, as will now be explained.

Assuming that the nuclear Hamiltonian is rotationally invariant, it commutes with the angular momentum $\vec{I}$, so it can be simultaneously diagonalized together with I_z and I^2. The energy eigenstates can therefore be taken to have a definite total angular momentum. Barring accidental degeneracy of the energy between different total angular momenta, the ground state must therefore have a definite total angular momentum. Thus the nuclear state can be written as $|\alpha I m_I\rangle$, where α represents other state labels not related to the total angular momentum in the center of mass frame.

Magnetic dipole moment

The magnetic dipole moment operator is a vector operator $\vec{\mu}$ with respect to the nuclear rotation generators $\vec{I}$. The Wigner-Eckart theorem tells us that the vector of matrix elements of $\vec{\mu}$ is parallel to that of the angular momentum $\vec{I}$ (a.k.a. the nuclear "spin"),

$$\langle m_I'|\vec{\mu}|m_I\rangle \propto \langle m_I'|\vec{I}|m_I\rangle, \tag{1}$$

where the αI labels of the states are omitted for notational brevity. In particular, the expectation value of the magnetic moment is aligned (or anti-aligned) with the spin. The 'value' of the magnetic moment is defined as the expectation value in the top state,

$$\mu := \langle m_I = I|\vec{\mu}|m_I = I\rangle. \tag{2}$$

Note that this vanishes for a spin-0 nucleus, because, having spin-0, the nucleus is rotationally invariant, so cannot define a direction for the expectation value of μ. Argued differently, the selection rule part of the Wigner-Eckart theorem tells us that, since $\vec{\mu}$ is a rank-1 tensor operator, this matrix element vanishes unless the representation I is included in the tensor product $1 \otimes I = (I+1) \oplus I \oplus |I-1|$. That is, it vanishes unless $I \geq |I-1|$, which holds for all $I \geq \frac{1}{2}$ but not for $I = 0$. So spin-0 nuclei have no magnetic dipole moment.

Electric dipole moment

Everything just said about the magnetic dipole moment applies also to the electric dipole moment (EDM) $\vec{d}$, however there is an additional symmetry consideration: $\vec{d}$ is invariant under time reversal T. This means there is a mismatch with the spin $\vec{I}$, which flips sign under T. If the Hamiltonian is invariant under time reversal, then this mismatch implies that the EDM vanishes. (I think this last statement should be plausible, but not completely obvious. It's tricky to formulate this sharply, because the time-reversal operator is anti-unitary, which requires delicate handling...)

The EDM of the neutron, nEDM, is constrained by measurements to be no larger than $3 \times 10^{-26} e$-cm. However, it should not be exactly zero. For one thing, the Hamiltonian of the standard model is *not* T (or CP[1]) invariant. For one thing, the phases in the quark mixing matrix violate T invariance, but this implies only a very small nEDM, of order $10^{-31} e$-cm, five orders of magnitude smaller than the current upper bound. However, there is another expected source of T violation, coming from QCD, and characterized by an angle θ. The current observational upper bound implies that this angle is smaller than 10^{-10} radians. The difficulty of explaining the smallness of this angle is called the *strong CP problem.* The leading idea for explaining it invokes a mechanism that in effect turns the value of this angle into an additional field, called the *QCD axion.* The axion is a dark matter candidate, but as yet has not been observed, nor has any nEDM. Several experiments on both fronts are ongoing.

Quadrupole moment

The Cartesian components of the electric quadrupole moment are defined by $Q^{ij} = \int \rho(x^i x^j - \frac{1}{3}\delta^{ij})dV$, where ρ is the charge density. Q^{ij} is a rank-2 irreducible tensor operator with respect to the nuclear rotation generators $\vec{I}$, with spherical components Q_{2q}, $q = 0, \pm 1, \pm 2$. The "electric quadrupole moment of the nucleus" can be defined as the expectation value of Q_{20} in the top state,

$$\text{electric quadrupole moment} = \langle \alpha II | Q_{20} | \alpha II \rangle \tag{3}$$

The selection rule tells us that this vanishes unless $I \geq |I - 2|$, which holds only for $I \geq 1$. A spin-$\frac{1}{2}$ or spin-0 nucleus (or nucleon) therefore cannot have an electric quadrupole moment. The same goes for a magnetic quadrupole moment. The hyperfine splitting of the ground state of alkalai atoms is unaffected by the nuclear electric quadrupole moment. Why? (*Answer*: By the Wigner-Eckart theorem, the $S_{1/2}$ state of the outer electron can produce no quadrupole electric field gradients at the nucleus.)

[1]If CPT symmetry is assumed to hold—which is implied by axioms of relativistic quantum field theory that so far at least seem to hold—then T invariance is equivalent to CP invariance.