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Angular momentum

Rotations and angular momentum

Rotations R in space are implemented on QM systems by unitary transformations,

$$U(R) = e^{-i\theta^i J_i/\hbar}, \tag{1}$$

where J_i are the hermitian generators of rotation. The J_i are also the angular momentum operators, and are conserved if the Hamiltonian is invariant under rotations. The rotation group structure implies the commutation relations,

$$[J_i, J_j] = i\hbar\epsilon^{ijk} J_k. \tag{2}$$

Unitary irreducible representations of the rotation group

Since $[J_z, J^2] = 0$,[1] we can simultaneously diagonalize J_z and J^2. Call the (normalized) eigenstates $|jm\rangle$, where

$$J_z|jm\rangle = m|jm\rangle, \qquad J^2|jm\rangle = j(j+1)|jm\rangle, \tag{3}$$

with $\hbar = 1$ from now on. To infer the consequences of the commutation relations, it's convenient to introduce the complex linear combinations

$$J_\pm := J_x \pm iJ_y, \tag{4}$$

in terms of which the commutation relations (2) take the form

$$[J_z, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_z. \tag{5}$$

Using these commutation relations it can be shown that the action of $J_\pm$ changes m by ± 1, with the following coefficients,[2]

$$J_\pm|jm\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle, \tag{6}$$

and that positivity of the norms of $J_\pm|jm\rangle$ implies that the possible values of j and m are

$$j = 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ 2, \ldots, \qquad m = j,\ j-1,\ j-2,\ \ldots, -j. \tag{7}$$

The representation with a given j is called the "spin-j" representation, and it is $2j+1$ dimensional. These representations are *irreducible*, in the sense that there is no subspace that is invariant (mapped into itself) under all rotations. This follows from (6), which implies that by acting with rotations we move through all the states. Note that the representation is an abstract structure, which can be realized by many different physical or mathematical systems.

[1] This means that the scalar J^2 is invariant under infinitesimal rotations about the z axis.

[2] The key identities are $J_-J_+ = J^2 - J_z(J_z+1)$ and $J_+J_- = J^2 - J_z(J_z-1)$, so that $\|J_+|jm\rangle\| = j(j+1) - m(m+1)$ and $\|J_-|jm\rangle\| = j(j+1) - m(m-1)$.

Examples

3d vectors V^i form the spin-1 representation. The tensor product of two vectors is a rank two tensor V^iW^j. More generally, a rank two tensor is a 3×3 array T^{ij}, transforming in the same way as a tensor product of vectors V^iW^j. The set of rank two tensors transforms into itself under rotations, but not irreducibly. Rather, the antisymmetric part is by itself irreducible, and three dimensional, hence it is another spin-1 representation. The symmetric part is reducible into the tracefree part, and the part proportional to the Kronecker delta δ^{ij} (the coeffcient being 1/3 the trace of the matrix). The trace part transforms under the spin-0 representation, i.e. it is invariant under rotations. The symmetric tracefree part transforms under the spin-2 representation: a symmetric tracefree tensor has 5 independent components, while for $j = 2$ we have $2j + 1 = 5$. Thus, we have just seen that the tensor product of two spin-1 representations is the sum of spin-2, spin-1, and spin-0 representations. This relation can be expressed with a sort of arithmetic of tensor products:

$$1 \otimes 1 = 2 \oplus 1 \oplus 0 \quad \longleftrightarrow \quad 3 \times 3 = 5 + 3 + 1. \tag{8}$$

The $\otimes$ symbol represents tensor product, and the $\oplus$ symbol represents direct sum of vector spaces. The arithmetic on the right shows the counting of vector space dimensions of the different representations. The simplest example is the product of two spin-1/2 representations:

$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0 \quad \longleftrightarrow \quad 2 \times 2 = 3 + 1. \tag{9}$$

That is, $\frac{1}{2} \otimes \frac{1}{2}$ decomposes into the direct sum of a triplet and a singlet.

Addition of angular momenta

The tensor product $j_1 \otimes j_2$ of any two representations is spanned by the product basis, $\{|j_1m_1\rangle|j_2m_2\rangle\}$. This decomposes into irreducible representations (irreps). To enumerate these, start with the "top" J_z state, $|j_1j_1\rangle|j_2j_2\rangle$, i.e. the state with the largest possible value of J_z, which is $j_1 + j_2$, and work down to lower values of J_z by applying the lowering operator J_-. At each step the result will be a linear combination of all the product states with $m_1 + m_2$ equal to a given value of the total J_z. When this process lands on the lowest possible J_z value, here $-(j_1 + j_2)$, it has filled out a spin-$(j_1 + j_2)$ representation. For example, using (6), we find the state below the top state is

$$J_-|j_1j_1\rangle|j_2j_2\rangle = \sqrt{2j_1}|j_1, j_1 - 1\rangle|j_2j_2\rangle + \sqrt{2j_2}|j_1j_1\rangle|j_2, j_2 - 1\rangle. \tag{10}$$

Next, construct the spin-$(j_1 + j_2 - 1)$ representation, by successively applying J_-, to the next highest top J_z state, which is the linear combination of the two next-to-top J_z states that is orthogonal to the linear combination (10) used already in building the spin-$(j_1 + j_2)$ representation. Repeating this process until all the states are used up, one obtains the decomposition

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \cdots \oplus |j_1 - j_2|. \tag{11}$$

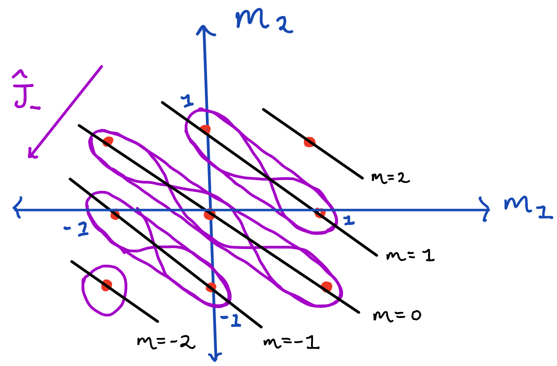

Figure 1: Each dot represents a state in the product basis, $|j_1 m_1\rangle|j_2 m_2\rangle$. States on each diagonal line have the same total $m = m_1 + m_2$. Acting repeatedly with J_-, starting with a top state with $m = j$, generates a basis for a spin-j representation. The purple curves looping around the states of a given m are meant to suggest the different linear combinations that occur in different j representations.

To see that the smallest representation in the sum is spin-$|j_1 - j_2|$, note that each of the states $|j_1 m_1\rangle$ must occur in every representation, since acting with J_+ and J_- will eventually introduce it. In particular, $|j_1 j_1\rangle$ must occur. If $j_1 \geq j_2$, the smallest *total* m that can occur in a product containing this state is in the product state $|j_1 j_1\rangle|j_2, -j_2\rangle$, in which case the total m is $j_1 - j_2$, hence the smallest representation has spin-$(j_1 - j_2)$. If instead $j_2 \geq j_1$, then the smallest representation has spin-$(j_2 - j_1)$. In general, then, the smallest is spin-$|j_1 - j_2|$. One can check that the total dimension $(2j_1 + 1)(2j_2 + 1)$ of $j_1 \otimes j_2$ is equal to the sum of $2j + 1$ over the j values stepping by integers from $j_1 + j_2$ down to $|j_1 - j_2|$.

For example, consider the p-wave electron states in an alkalai atom, with orbital angular momentum $\ell = 1$. These transform under the spin-1 representation, while the spin of the electron transforms under the spin-1/2 representation. The Hilbert space for these orbital and spin degrees of freedom is the tensor product of the two, $1 \otimes \frac{1}{2}$, which decomposes into the sum of two irreps,

$$1 \otimes \tfrac{1}{2} = \tfrac{3}{2} \oplus \tfrac{1}{2} \longleftrightarrow \quad 3 \times 2 = 4 + 2 \tag{12}$$

For another example, consider four spin-$\frac{1}{2}$ systems:

$$\tfrac{1}{2} \otimes \tfrac{1}{2} \otimes \tfrac{1}{2} \otimes \tfrac{1}{2} = (1 \oplus 0) \otimes (1 \oplus 0) \tag{13}$$
$$= (1 \otimes 1) \oplus 1 \oplus 1 \oplus 0 \tag{14}$$
$$= 2 \oplus 1 \oplus 0 \oplus 1 \oplus 1 \oplus 0. \tag{15}$$

So there is one spin-2 irrep, three triplets, and two singlets. The singlets are invariant under all rotations.

Clebsch-Gordan coefficients

The identity operator on the product $j_1 \otimes j_2$ can be expanded in $\{|j_1 m_1\rangle |j_2 m_2\rangle\}$ states, or in $|jm\rangle$ states:

$$I_{j_1 \otimes j_2} = \sum_{m1,\, m2} |m_1 m_2\rangle\langle m_1 m_2| = \sum_{j\, m} |jm\rangle\langle jm|. \tag{16}$$

Here I use the notational abbreviation $|m_1 m_2\rangle := |j_1 m_1\rangle|j_2 m_2\rangle$, suppressing the j_1 and j_2 labels since they are the same for all the states. Applying the identity in the form of the first of these sums yields

$$|jm\rangle = \sum_{m_1+m_2=m} |m_1 m_2\rangle\langle m_1 m_2|jm\rangle. \tag{17}$$

Similarly, applying the identity in the form of the second sum in (16) yields

$$|m_1 m_2\rangle = \sum_j |jm\rangle\langle jm|m_1 m_2\rangle, \tag{18}$$

with $m = m_1 + m_2$ (since otherwise the inner product vanishes). The inner products that serve as the expansion coefficients are called *Clebsch-Gordan (CG) coefficients.* The decomposition into the irreps discussed above introduces only algebraic functions involving the square root coefficients appearing in (6). Therefore the CG coefficients can always be taken to be real, and we have (restoring the explicit $j_1 j_2$ dependence)

$$\langle j_1 m_1 j_2 m_2|jm\rangle = \langle jm|j_1 m_1 j_2 m_2\rangle^* = \langle jm|j_1 m_1 j_2 m_2\rangle. \tag{19}$$

There remains a sign ambiguity, which is typically fixed by requiring that the coefficient of $|m_1 = j_1\rangle|m_2 = j - j_1\rangle$ in the the expansion of the top state $|jj\rangle$ of the spin-j representation is positive, i.e. $\langle j_1, j - j_1|jj\rangle > 0$.

The CG coefficients can be found in many ways: brute force, Mathematica: ClebschGordan[$\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\}$], tables, recursion relations, a projection operator method, and, amazingly enough, using a *closed form* formula found by Wigner. The formula was given in a more symmetrical form by Racah, but it's too complicated to be usable. See (106.14) of Landau & Lifshitz, QM.