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Born approximation validity conditions

Integral form of time-independent Schrodinger equation for an incoming plane wave in a potential V:

$$\psi(x) = e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{m}{2\pi\hbar^2}\int d^3x'\,\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}V(\mathbf{x}')\psi(\mathbf{x}') \tag{1}$$

The (first) Born approximation consists of replacing $\psi(\mathbf{x}')$ in the integrand by $e^{i\mathbf{k}\cdot\mathbf{x}'}$:

$$\psi^{FBA}(x) = e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{m}{2\pi\hbar^2}\int d^3x'\,\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}V(\mathbf{x}')e^{i\mathbf{k}\cdot\mathbf{x}'}. \tag{2}$$

A sufficient condition for this to be a good approximation is that the difference between $\psi(\mathbf{x})$ and $e^{i\mathbf{k}\cdot\mathbf{x}}$ be small in the region $\mathcal{R}$ where the integral receives important contributions; i.e. if for all $\mathbf{x}\in\mathcal{R}$,

$$\frac{m}{2\pi\hbar^2}\left|\int d^3x'\,\frac{e^{i(k|\mathbf{x}'-\mathbf{x}|+\mathbf{k}\cdot\mathbf{x}')}}{|\mathbf{x}'-\mathbf{x}|}V(\mathbf{x}')\right| \ll 1. \tag{3}$$

In terms of $\mathbf{r}=\mathbf{x}'-\mathbf{x}$, and with $\mathbf{x}$ chosen as the coordinate origin, this can be written more simply as

$$\frac{m}{2\pi\hbar^2}\left|\int d^3r\,\frac{e^{i(kr+\mathbf{k}\cdot\mathbf{r})}}{r}V(\mathbf{r})\right| \ll 1. \tag{4}$$

Since oscillations of the exponential only make the integral smaller, a sufficient condition for (4) is

$$\frac{m}{2\pi\hbar^2}\left|\int d^3r\,\frac{1}{r}V(\mathbf{r})\right| \ll 1. \tag{5}$$

Note that this is a k-independent condition. If the potential is characterized by a largest absolute value $V_{\max}$, and is negligible outside some radius a. Then (5) is guaranteed if

$$\frac{V_{\max}}{\hbar^2/ma^2} \ll 1 \qquad \textit{weakness condition} \tag{6}$$

That is, the potential energy should be much smaller than the kinetic energy associated with the particle being localized in the potential. Another way to characterize this is that the scattering amplitude should be much smaller than a, implying that that the cross section σ should be much smaller than the geometric cross-section πa^2. A similar condition can be derived even if $V(\mathbf{r})$ has no maximum, as long as the integrand has a maximum. For instance, although $V(r)=\frac{1}{r}e^{-r/a}$ has no maximum, $rV(r)$ does have one.

In the high energy case $ka \gg 1$, oscillations of the exponential make the integral smaller, so the Born approximation can be valid even if the potential is not "weak" in the sense of (7). The phase $kr+\mathbf{k}\cdot\mathbf{r}$ is stationary (and vanishes) for $\mathbf{r}$ opposite to $\mathbf{k}$, and is not oscillating rapidly only in the solid angle where $ka(1-\cos\theta)\lesssim 1$, i.e. where $\delta\theta^2 \lesssim 1/ka$. This means the integral is of order $1/ka$ times what it is in the long wavelength case $ka \ll 1$. Thus a sufficient condition for validity is

$$\frac{V_{\max}}{\hbar^2/ma^2} \ll ka \qquad \textit{high energy condition} \tag{7}$$

This is similar to, but less stringent by the factor 4π than, the validity condition $\int V\,dt \ll \hbar$ for treating the scattering by first order time-dependent perturbation theory. I'm not completely sure about the 4π. It would be good to test this on an example where the exact result is known.