**HW#9**—Phys623—Spring 2018 Prof. Ted Jacobson
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**1.** *The 21 cm line*

**(a)** Which type of multipole interaction dominates the $F = 1$ to $F = 0$ decay of the hyperfine of the ground state of hydrogen (a.k.a. the 21 cm line)? Why?

**(b)** Compute the decay rate for each of the three starting $m_F$ values. What is the lifetime in years? (You need not rederive the general formula for the decay rate that we derived in class, but you should evaluate the relevant matrix elements, and polarization and wavevector sums.) [*Hints*: i) It's more than a million and less than a billion... ii) The rate is the same for all three $m_F$. This is a consequence of spherical symmetry, but doesn't quite seem obvious to me, because the states with different $m$ values are not related by rotations. Do you see a slick argument for this equality? iii) The decay rate includes a summation over the polarizations and an integral over the wavevector direction. The combined sum and integral is $\int d\Omega \sum_\lambda (k \times \epsilon)^a (k \times \epsilon^*)^b$. An easy way to evaluate this is to argue that the result must be proportional to $\delta^{ab}$, and then to evaluate the coefficient by taking the trace of both sides. Alternatively, first do the polarization sum, using a vector index notation.]

**(c)** Suppose the initial state has $m_F = 1$. Calculate the polarization of the emitted photon if its wavevector is in the direction (i) $\hat{z}$, (ii) $-\hat{z}$, (iii) $\hat{x}$. [*Comment*: The two cases with $\hat{k}$ along the $z$ axis can be inferred directly from angular momentum conservation, but show that this follows from your expression for the decay rate. For $k = k\hat{x}$, it's not clear to me how to use the conservation law.]

**2.** This problem concerns spontaneous dipole transitions of a hydrogen atom.

**(a)** Make an argument using dimensional analysis and basic physics to determine how the total rate $\Gamma$ for an electric dipole transition of hydrogen depends upon $e$, $c$, $\hbar$, the frequency $\omega$ of the emitted photon, and the dipole matrix element $\mathbf{d} = \langle f|\mathbf{r}|i\rangle$. **Explain your logic**.

**(b)** Use your result from part 2a to make an order of magnitude estimate of the rate for the transition $2p \to 1s$. Next, compare this with the rate for the transition $2s_{1/2} \to 2p_{1/2}$. (The Lamb shift is of order $10^{-1}\alpha^3 \ln \alpha$ Rydbergs.)

*Possibly useful information*:
$\alpha = e^2/\hbar c \simeq 1/137$
$\hbar \simeq 2/3$ eV-fs (1 fs = $10^{-15}$ s)
1 Rydberg = 13.6 eV

**3.** *Qualifier, Problem II-4, Spring 1998*

This problem concerns spontaneous decay of an excited state of hydrogen with the emission of one photon with momentum $\hbar\mathbf{k}$ and polarization vector $\epsilon$ with $\mathbf{k}\cdot\epsilon = 0$. The probability amplitude for such a transition between atomic states $|i\rangle$ and $|f\rangle$ is proportional to the dimensionless matrix element $\langle f|(M_{orb} + M_{spin})|i\rangle$ with

$$\langle f|M_{orb}|i\rangle = \frac{1}{mc}\,\epsilon^* \cdot \langle f|\mathbf{p}\,e^{-i\mathbf{k}\cdot\mathbf{r}}|i\rangle \tag{1}$$

$$\langle f|M_{spin}|i\rangle = \frac{i}{mc}\,(\mathbf{k}\times\epsilon^*) \cdot \langle f|\mathbf{S}\,e^{-i\mathbf{k}\cdot\mathbf{r}}|i\rangle \tag{2}$$

where $\mathbf{r}$ is the position vector of the electron, $\mathbf{p}$ is the momentum, and $\mathbf{S}$ is the spin.

**(a)** Show that, for the $2p \to 1s$ transition, $\langle 1s|(M_{orb}+M_{spin})|2p\rangle$ is of order $\alpha$, where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant. (This leads to a lifetime of order $10^{-9}$ s for the $2p$ state.)

The $2s$ state has a much longer lifetime, of order $1/7$ s, because many types of transitions are forbidden. The remaining parts of this problem explore this phenomenon.

**(b)** Show that $\langle 1s|M_{orb}|2s\rangle = 0$.

**(c)** Consider the expansion of the spin transition matrix element $\langle f|M_{spin}|i\rangle = \sum_{n=0}^{\infty}(1/n!)\langle f|M^{(n)}_{spin}|i\rangle$, with $\langle f|M^{(n)}_{spin}|i\rangle = \frac{i}{mc}\,(\mathbf{k}\times\epsilon^*)\cdot\langle f|\mathbf{S}\,(-i\mathbf{k}\cdot\mathbf{r})^n|i\rangle$. Show that $\langle 1s|M^{(n)}_{spin}|2s\rangle$ is zero for $n = 0, 1$ but non-zero for $n = 2$.

**(d)** Estimate the order of magnitude of $\langle 1s|M^{(2)}_{spin}|2s\rangle$ in powers of $\alpha$.

**(e)** Using parts 3a and 3d estimate the lifetime of the 2s state assuming the transition proceeds via the matrix element $\langle 1s|M^{(2)}_{spin}|2s\rangle$. (*Answer*: I think it's $\sim 10^4$ s.)

**(f)** The actual lifetime (1/7 s) is much shorter than the lifetime you should have obtained in part 3e. Can you think of a reason? (It is *not* that the atom first decays via the allowed $2s_{1/2} \to 2p_{1/2}$ dipole transition. Why?)

**4.** *Optional challenge for extra credit*: Littlejohn 42.1, $2s \to 1s$ two photon decay in hydrogen.