HW#8—Phys623—Spring 2018

Prof. Ted Jacobson
PSC 3151, (301)405-6020
jacobson@physics.umd.edu

1. *Selection rules for atomic transitions*: Consider the matrix elements of the form

$$\langle \gamma' J' M_{J'} L' S' | \hat{T} | \gamma J M_J L S \rangle$$

where the states describe an atom with angular momentum quantum numbers $JLSM_J$, and remaining labels γ needed to specify a state, and the operator $\hat{T}$ is either the electric dipole, orbital magnetic dipole, spin magnetic dipole, or electric quadrupole transition operator. For the purposes of this problem, you only need to know that these operators are c-numbers times $\sum x^i$, $\sum L^i$, $\sum S^i$ and $\sum x^i x^j - \frac{1}{3}x^2\delta^{ij}$, respectively, where i and j are vector indices, the sums are over the electrons in the atom, and the letters have the usual meanings. The first three of these operators are irreducible tensor operators of rank 1, and the last is of rank 2. They are all tensor operators wrt $\vec{J}$, $\vec{L}$, and $\vec{S}$ (but the spin operator is scalar wrt $\vec{L}$ and the orbital operators are scalars wrt $\vec{S}$). Determine in each of these cases the joint selection rules for parity, J, M_J, and L. Make a table displaying the selection rules, and explain very briefly your reasoning.

2. *Quantum fluctuations of the electromagnetic field*

Schwabl's (16.47a) together with (16.49) gives the electric field operator.

(a) Calculate the vacuum expectation value ("vev") of the electric field operator, $\langle 0 | E_i(\vec{x}, t) | 0 \rangle$.

(b) Show that (in the infinite volume limit) the vev of the product of two electric field operators at different positions at equal times takes the form

$$\langle 0 | E^i(\vec{r}, t) E^j(0, t) | 0 \rangle = (a\,\delta^{ij} + b\,\hat{r}^i \hat{r}^j)\hbar c / r^4, \tag{1}$$

for some numerical constants a and b. This shows that the field fluctuations at two spacelike separated points are correlated, and that the variance diverges as the two points approach each other.

(c) Show that Gauss' law implies that $b = -2a$. This implies that the components along $\hat{r}^i$ have equal and opposite correlation to those perpendicular to $\hat{r}^i$.

(d) Show that $a = -4/\pi$. (In the infinite volume limit the sum over wavevectors becomes an integral. Regulate the integral by inserting $e^{-\epsilon k}$ into the integrand, evaluate the integral, and then take the limit $\epsilon \to 0$.) Thus the components along $\hat{r}^i$ have positive correlation while those perpendicular have negative correlation.

3. To better understand the quantization of the electromagnetic field (or even a scalar field), it's helpful to reformulate the classical description in a way that matches a harmonic oscillator. Since the plane wave Fourier amplitudes of the field are complex, the comparison with the oscillator is facilitated by first expressing the oscillator using a complex phase space coordinate. The Hamiltonian for a harmonic oscillator is

$$H = (p^2 + x^2)/2 = \alpha^*\alpha, \quad \text{where} \quad \alpha = (x + ip)/\sqrt{2}, \tag{2}$$

in units with $m = \omega = 1$. The Poisson bracket is $\{\alpha, \alpha^*\} = -i$. In preparation for quantization, let's make α dimensionless by including a factor $1/\sqrt{\hbar\omega}$ in the definition. In arbitrary units, with $x_0 := \sqrt{\hbar/m\omega}$, we thus have

$$H = \hbar\omega\,\alpha^*\alpha, \qquad \alpha = (x/x_0 + i\,x_0 p/\hbar)/\sqrt{2}, \qquad \{\alpha, \alpha^*\} = -i/\hbar. \tag{3}$$

Upon canonical quantization, α becomes the lowering operator a, and the Poisson bracket becomes the commutation relation $[a, a^\dagger] = 1$.

Similarly, for the electromagnetic field we can trade Schwabl's field amplitude A_k (16.47b) and it's conjugate momentum (which is proportional to $\dot{A}_{-k}$) for a complex phase space coordinate,

$$C_k := (A_k + i\,\dot{A}_k/\omega)/2, \tag{4}$$

where $\omega := ck$. (For notational simplicity I'm writing the vector index $\mathbf{k}$ simply as k, and not indicating the vector nature of A_k and C_k.) Using the reality condition $A_k^* = A_{-k}$, it follows from (4) that

$$A_k = C_k + C_{-k}^*. \tag{5}$$

(a) The $|\dot{A}_k|^2$ term in Schwabl's Hamiltonian (16.48) also equal to the time derivative term in the Lagrangian, and it arises as $\dot{A}_k\dot{A}_{-k}$. Use this to show that the canonical momentum conjugate to A_k is given by

$$\Pi_k = \frac{V}{4\pi c^2}\dot{A}_{-k}. \tag{6}$$

(b) Use the Poisson bracket relations $\{A_k, \Pi_l\} = \delta_{kl}$ to deduce the brackets

$$\{C_k, C_l^*\} = -i\frac{2\pi c}{Vk}\,\delta_{kl}. \tag{7}$$

In terms of the rescaled coordinates $\alpha_k := C_k/\sqrt{2\pi\hbar c/Vk}$, (7) is equivalent to

$$\{\alpha_k, \alpha_l^*\} = -\frac{i}{\hbar}\delta_{kl}. \tag{8}$$

Compare with the harmonic oscillator in order to justify the expression (16.49) for the vector potential field operator.

(c) Show that in terms of α_k the Hamiltonian (16.48) takes the form

$$H_{\text{rad}} = \sum_k \hbar\omega\,\alpha_k^*\alpha_k, \tag{9}$$

and that upon quantization this leads to the Hamiltonian operator (16.51a).