HW#7—Phys623—Spring 2018 Prof. Ted Jacobson
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1. Consider the Hamiltonian $H(t) = H_0 + \alpha(t)V$, where (generically) $[H_0, V] \neq 0$, and $\alpha(t)$ is a real function that vanishes if $|t| > \tau$.

(a) In the limit $\tau \to 0$, with $\gamma \equiv \int dt\, \alpha(t)$ fixed, the effect of V on the evolution is "impulsive". Find the exact state $|\psi(0_+)\rangle$ just after the impulse in terms of the state $|\psi(0_-)\rangle$ just before the impulse. (*Hint*: Make sure the evolution is unitary. Because the action of H_0 on $|\psi(0_-)\rangle$ is finite, H_0 does not affect the evolution over an infinitesimal time interval, in contrast to $\alpha(t)V$, since $\alpha(t)$ becomes a delta function in the impulsive limit.)

(b) Write the first order approximation for the state $|\psi(t)\rangle$ in terms of $|\psi(-\tau)\rangle$ using time-dependent perturbation theory.

(c) Take the impulsive limit of the first order perturbative result. Under what conditions does it give a good approximation to the exact impulsive result?

2. Consider the Hamiltonian given by $H(t) = H_0 + e^{\eta t} V_0$ for $t < 0$, and by $H(t) = H_0 + V_0$ for $t \geq 0$. Assume H_0 has discrete spectrum, and suppose the initial state at $t \to -\infty$ is an eigenstate $|n\rangle$ of H_0, with $H_0|n\rangle = E_n|n\rangle$.

(a) Show that, in the limit $\eta \to 0$, the first order time-dependent shift of the state, evaluated at $t = 0$, agrees with the first order shift of the state in time-independent perturbation theory (with V_0 as the perturbation). Restrict here to the eigenkets $|m\rangle$ of H_0 that are orthogonal to $|n\rangle$. (In fact, the exact time dependent solution coincides with an exact eigenstate of $H_0 + V_0$ in this limit. This is an instance of the *adiabatic theorem*.)

(b) How small must η be to ensure that the first order time-dependent shift of the state at $t = 0$ agrees to within 1% with the first order shift of the state in time-independent perturbation theory? It depends on the component $|m\rangle$ of the state, so give an answer for each m.

(c) The shift of the component parallel to the original eigenket $|n\rangle$ diverges as $1/\eta$ in the $\eta \to 0$ limit. Show that this can be understood as a result of the perturbation of the time-dependent phase of that ket. To this end, consider evolution from some finite $t = t_1 < 0$ to $t = 0$, in the limit $\eta \to 0$. (*Hint*: Time-independent perturbation theory will give you (to first order) the energy shift, with which you can find the time-dependent change of the phase.)

3. The time evolution of an harmonic oscillator acted on by any time dependent external force $F(t)$ can be found exactly in quantum mechanics, just as it can in classical mechanics. In this problem you'll learn how that works, and apply it to find the state at time t, if the system starts in the ground state at $t = 0$.

The force $F(t)$ is derived from the potential $-F(t)x = -f(t)(a + a^\dagger)$, where a and $a^\dagger$ are the ladder operators for the harmonic oscillator, and $f(t) = F(t)x_0/\sqrt{2}$. Here

$x_0 = \sqrt{\hbar/m\omega}$, where m and ω are the mass and frequency of the oscillator. The Hamiltonian is thus

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}) - f(t)(a + a^\dagger). \tag{1}$$

At this stage let's set $\hbar = 1$ by a units choice.

(a) Verify that, for a quantum system with Hamiltonian $H(t)$, the solution to the Schrodinger equation is given by $|\psi(t)\rangle = U(t,t')|\psi(t')\rangle$, where the time evolution operator $U(t,t')$ is defined to satisfy $i\partial_t U(t,t') = H(t)U(t,t')$, with the boundary condition $U(t,t) = I$.

(b) In the Heisenberg picture, the lowering operator evolves as $a_H(t) = U^\dagger a U$, where $U \equiv U(t,0)$, and $a = a_H(0)$ is the Schrodinger picture lowering operator. Show that a_H satisfies the differential equation $\dot{a}_H = -i\omega a_H + if$, and show that the solution is given by

$$a_H(t) = e^{-i\omega t}a + \alpha(t), \quad \alpha(t) = ie^{-i\omega t}\int_0^t dt' e^{i\omega t'} f(t'). \tag{2}$$

This is basically the complete solution to the dynamics, since any observable in the Heisenberg picture can be constructed from $a_H(t)$.

(c) Using the previous result, show that if the oscillator starts out in the ground state $|0\rangle$ at $t = 0$, the state thereafter satisfies $a|\psi(t)\rangle = \alpha(t)|\psi(t)\rangle$. That is, the ground state evolves to the "coherent state" $|\alpha(t)\rangle$.

(d) In case you haven't yet come across coherent states, or have forgotten how they work, show that the normalized coherent state defined by the condition $a|\alpha\rangle = \alpha|\alpha\rangle$ is given by

$$|\alpha\rangle := e^{-|\alpha|^2/2}\sum_n \frac{1}{\sqrt{n!}}\alpha^n|n\rangle, \tag{3}$$

where $|n\rangle$ is the number eigenstate.

(e) Find the probability $P_n(t)$ for the oscillator to be in its nth level at time t.

(f) Find $P_n(t)$ using first order time dependent perturbation theory.

(g) Under what conditions does first order perturbation theory give an accurate result for $P_n(t)$?

continued...

4. *Scattering with free vs. non-free final states* A particle of mass m moves in a one-dimensional attractive potential $U(x) = -\gamma\delta(x)$, where $\delta(x)$ is the Dirac delta-function, and $\gamma > 0$. Use periodic boundary conditions with $x = \pm L$ identified, with $L \to \infty$. Since you're taking L to infinity, you need only retain the leading order contributions in $1/L$.[1]

(a) Find the wave function and the energy E_0 of the bound state. What is the parity of the wave function with respect to the operation $x \to -x$?

(b) Find the wave functions and the energies of the unbound states. Choose the wave functions to be symmetric or antisymmetric with respect to the parity operation $x \to -x$. [*Hint*: The symmetric states look like free states with a kink at $x = 0$, so can be written in the form $\psi_n(x) = \frac{1}{\sqrt{L}} \cos(k_n|x| + \phi_n)$.]

At time $t < 0$, the particle is in the ground state of the potential. At time $t > 0$, a small oscillating potential

$$V(t) = Ax^2 \sin(\omega t) \tag{4}$$

of frequency $\omega > |E_0|/\hbar$ is turned on.

(c) Which matrix elements of the perturbation (4) between the ground state and the symmetric or antisymmetric unbound states vanish because of the parity selection rule? Calculate the nonvanishing matrix elements.

(d) Using Fermi's golden rule, calculate the probability of transition of the particle from the ground state to an unbound state per unit time.

(e) Discuss and interpret the behavior of the ejection rate as ω approaches ∞, and as $\hbar\omega$ approaches $|E_0|$. There is a frequency $\omega > |E_0|/\hbar$ for which the transition rate vanishes. What is this frequency, and why does the (first order) rate vanish for this particular frequency?

(f) How would the results change if, for the final states, you used free particle states rather than the exact unbound states? Show that there is a large difference in behavior at threshold $\omega = |E_0|/\hbar$, but a small difference when $\omega \gg |E_0|/\hbar$.

[1] Note that we can choose units with $\hbar = m = \gamma = 1$. That simplifies the equations, but sacrifices the ability to catch errors using dimensional analysis. You may choose whether or not to live dangerously. If you do, you should double check yourself along the way, and restore the dimensionful constants at the end. I like the following method for restoring the constants. First, note that γ has dimensions of energy $\times$ length, so $l_u = \hbar^2/m\gamma$ has dimensions of length, $\omega_u = \gamma/\hbar l_u = m\gamma^2/\hbar^3$ has dimensions of frequency, and $A_u = \gamma/l_u^3 = m^3\gamma^4/\hbar^6$ has the dimensions of A, i.e. energy/length^2. Your result for the rate will be some function of the remaining constants, $\Gamma(\omega, A)$. To restore the other constants—i.e. to write the result in a form that holds in any system of units—you simply replace $\Gamma(\omega, A)$ by $\omega_u\Gamma(\omega/\omega_u, A/A_u)$. This is correct because it has the correct dimension (inverse time), and agrees with your result when using the adopted units.