HW#3—Phys623—Spring 2018

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1. The ground state of the hydrogen atom is split into two hyperfine states separated by 1.42 GHz. What is the hyperfine splitting in the deuterium atom? The respective magnetic moments are $\mu = 2.8\mu_N$ and $\mu_d = 0.86\mu_N$, where μ_N is the nuclear magneton, and the deuteron has spin 1. (*Answer*: 0.33 GHz.)

2. The deuteron is the unique bound state of a neutron and a proton.

(a) Using only the facts that (i) the neutron and proton are both spin-1/2 particles, and (ii) the deuteron has total angular momentum $\hbar$ ("spin 1"), what would be the possible values of ${}^{2S+1}L_J$ for the deuteron? (Here S, L, and J are the spin, orbital and total angular momentum of the bound state, respectively.)

(b) Which combinations of the ${}^{2S+1}L_J$ found in part 2a could occur in the deuteron, given that parity is a symmetry of the nuclear hamiltonian? Justify your answer.

3. Since the strong interaction is short ranged, there must be a large S-wave component in the deuteron wave function, and in fact this wave function is a superposition $a\,{}^3S_1 + b\,{}^3D_1$. In this problem you will determine the fraction $|b|^2$ of the D-wave component by calculating the magnetic moment and comparing with the observed value. The magnetic moment operator for the deuteron is

$$\boldsymbol{\mu} = \frac{\mu_N}{\hbar}(\mathbf{L}_p + g_p\mathbf{S}_p + g_n\mathbf{S}_n) \tag{1}$$

where $g_p = 5.587$ and $g_n = -3.826$. The magnetic moment *value* is the expectation value of μ_z in the state with maximal total angular momentum in the z-direction.

(a) Show that $\mathbf{L}_p = \frac{m_n}{m_p+m_n}\mathbf{L} = 0.50034\,\mathbf{L}$. For the rest of this problem you may neglect the neutron-proton mass difference.

(b) Evaluate the magnetic moment μ_d of the deuteron in terms of $|b|^2$, g_p and g_n.

(c) Set your result for μ_d equal to the observed value $0.85735\mu_N$ and solve for $|b|^2$. (*Answer*: $|b|^2 \approx 0.04$.)

Hint: To evaluate the contribution from the expectation value in the ${}^3D_1$-state, you can either express this state as a linear combination of products of $|Lm_L\rangle$ and $|Sm_S\rangle$ eigenstates using Clebsch-Gordan coefficients, or use the *projection theorem* for vector operators. (For good practice, do it both ways!) For the latter method, I think you'll want to use the result of problem 4 to show that $\mathbf{S}_p$ and $\mathbf{S}_n$ have equal expectation values in an eigenstate of $\mathbf{S}^2$.

4. Consider addition of any two equal angular momenta, $j_1 = j_2$. Show that in any irreducible representation j within the tensor product $j_1 \otimes j_2$, the matrix elements of $\vec{J}_1$ are equal to those of $\vec{J}_2$,

$$\langle jm_j'|\vec{J}_1|jm_j\rangle = \langle jm_j'|\vec{J}_2|jm_j\rangle = \frac{1}{2}\langle jm_j'|\vec{J}|jm_j\rangle. \tag{2}$$

This can be shown using the projection theorem for vector operators (which is explained in my notes on the Wigner-Eckart theorem), or by an argument using the symmetry under interchange of m_1 and m_2 (or perhaps another way).

5. The Weyl equation for massless, spin-1/2 particles is

$$i\partial_t \chi = \pm\vec{\sigma}\cdot\vec{p}\,\chi,$$

where $\vec{\sigma}$ is the vector of Pauli matrices, χ is a two-component spinor, and $\vec{p}$ is the usual quantum mechanical momentum operator. It is the time evolution equation for a quantum system with Hamiltonian $H = \pm\vec{\sigma}\cdot\vec{p}$. The two possible signs correspond to the "chirality" of the massless spin-1/2 particle, called right-handed (+) and left-handed (−).

(a) Use dimensional analysis to restore the factors of $\hbar$ and c in the Weyl equation (but after this part set $\hbar = c = 1$).

(b) Establish the following:

i. The Hamiltonian is translation invariant, and momentum conserving.

ii. The Hamiltonian is invariant under time reversal, but not under parity.[1]

iii. The "helicity" $\frac{1}{2}\vec{\sigma}\cdot\hat{p}$ (spin along the momentum) is conserved in time.

iv. An eigenstate of momentum and helicity is an energy eigenstate with energy E satisfying $E = \pm|\vec{p}|$. Energy eigenstates of right (left)-handed Weyl particles with positive (negative) helicity have positive energy.

v. The Heisenberg spin operator satisfies $d\vec{S}/dt = \mp 2\,\vec{p}\times\vec{S}$. The components of spin orthogonal to the momentum are therefore *not* conserved: the spin precesses around the momentum direction.

vi. The Heisenberg velocity operator $d\vec{x}/dt$ is $\pm\vec{\sigma}$. The component of velocity along the momentum is therefore the speed of light, while the components orthogonal to the momentum precess around the momentum direction.

(c) The Dirac equation consists of a pair of Weyl equations, for right and left chirality spinors R and L, coupled by terms that evolve R into L and vice versa:

$$i\partial_t R = \vec{\sigma}\cdot\vec{p}\,R + mL, \qquad i\partial_t L = -\vec{\sigma}\cdot\vec{p}\,L + mR.$$

i. Show that $H^2 = p^2 + m^2$, where H is the full Hamiltonian acting on the 4-component wave function $\begin{pmatrix} R \\ L \end{pmatrix}$. This means that an energy and momentum eigenstate satisfies $E^2 = p^2 + m^2$, so that m is evidently the mass of the particle.

ii. Find the positive energy eigenstates with zero momentum. Show that both signs of energy occur.

[1] You may answer this just by invoking how momentum and spin transform. However, for the case of time reversal, you can also show explicitly that the Hamiltonian commutes with the time reversal operator acting on spinors, $\chi \to i\sigma_2\chi^*$.