HW#1—Phys623—Spring 2018

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1. Problem 11.1, *Schwabl.* (Use units with $\hbar = m = \omega = 1$, and use Schwabl's hint. Note ax^3 should be bx^3 in the statement of the problem.) To reduce the labor, *solve this problem only for the ground state.* Add parts: (b) Find the first order perturbation to the ground state vector. (c) Find the first order perturbation to the expectation value of x in the ground state, and explain its sign qualitatively in terms of the shape of the potential function.

2. Problem 11.3, *Schwabl.* (N.B. also $\omega = 1$. Also, use ladder operators to save labor.) This problem can be solved either using degenerate perturbation theory with the x and y oscillators, or using nondegenerate perturbation theory by decoupling the system into two anharmonic oscillators, with generalized coordinates $u = (x+y)/\sqrt{2}$ and $v = (x-y)/\sqrt{2}$. Solve it both ways, and make sure your answers for the perturbed energy levels agree. (You need not find the perturbed states.)

3. Consider the Hamiltonian $H = H_0 + V$ of a three state system, with

$$H_0 = |3\rangle\langle 3|, \qquad V = ve^{i\phi}\Big(|1\rangle + |2\rangle\Big)\langle 3| + \text{h.c.} \tag{1}$$

The eigenvalue 0 of H_0 is doubly degenerate, and the perturbation does not break this degeneracy at first order, so this is an example where one needs to impose the second order secular equation in order to obtain the correct second order energy shifts.

As you can check by hand (but, to save labor, need not), the exact eigenvalues are $0, \lambda_\pm$, with corresponding normalized eigenvectors

$$(1,-1,0)/\sqrt{2} \quad \text{and} \quad a_\pm(ve^{i\phi}, ve^{i\phi}, \lambda_\pm), \tag{2}$$

where

$$\lambda_\pm = (1 \pm \sqrt{1+8v^2})/2 \quad \text{and} \quad a_\pm = 1/\sqrt{2v^2 + \lambda_\pm^2}. \tag{3}$$

Expanding in small v (assumed > 0), and keeping up to quadratic terms in v, one can (carefully) show that the nonzero eigenvalues and corresponding (normalized to $O(v^2)$) eigenvectors become

$$\lambda_+ \;=\; 1 + 2v^2, \quad (ve^{i\phi}, ve^{i\phi}, 1 - v^2) \tag{4}$$

$$\lambda_- \;=\; -2v^2, \quad \Big((1-v^2)e^{i\phi}, (1-v^2)e^{i\phi}, -2v\Big)/\sqrt{2} \tag{5}$$

(a) Use degenerate perturbation theory (with V as the perturbation) to find the first and second order energy shifts $E^{(1)}$ and $E^{(2)}$ for the three eigenvalues, and check that they agree with the expansions given above.

(b) Find the eigenvectors of the second order secular equation and compare with the $v \to 0$ limit of the exact eigenvectors. They should agree.

4. Consider a two-state quantum system described by the Hamiltonian

$$H = \begin{pmatrix} E+U & \Delta e^{i\phi} \\ \Delta e^{-i\phi} & E-U \end{pmatrix}, \tag{6}$$

with $E > 0$, U, $\Delta > 0$, and ϕ all real. This is the most general hermitian 2×2 matrix.

(a) Find the exact eigenvalues and eigenvectors of H.

(b) Sketch the eigenvalues as functions of U in the range $U \ll -\Delta$ to $U \gg \Delta$. Notice that the energy levels "repel" in the region $U \approx 0$ where they would cross if Δ were zero.

(c) Expand the exact eigenvalues to lowest nonvanishing order in U/Δ when $|U| \ll \Delta$.

(d) Considering the Δ terms of the Hamiltonian (6) as a perturbation, compute the first and second order energy level shifts, and the first order correction to the energy eigenstates, using non-degenerate perturbation theory (assuming $U \neq 0$).

(e) The approximate eigenvalues of parts (4c) and (4d) don't agree when $|U| \ll \Delta$. Explain why non-degenerate perturbation theory does not give good results even though the unperturbed eigenvalues are non-degenerate when $U \neq 0$. What should be done in this case to obtain a good approximation using perturbation theory?