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# Irreducible tensor operators and the Wigner-Eckart theorem

**1.** An *irreducible tensor operator* of order $k = 0, 1/2, 1, 3/2, \ldots$ is a collection of operators $T_{kq}$, $q = k, k-1, \ldots, -k$, that transforms under rotations like the spherical harmonics $Y_{kq}(\theta, \phi)$, considered as multiplication operators, i.e.

$$[J_z, T_{kq}] \;=\; q\, T_{kq} \tag{1}$$

$$[J_\pm, T_{kq}] \;=\; \sqrt{k(k+1) - q(q\pm 1)}\, T_{k,q\pm 1} \tag{2}$$

where it is understood that $T_{kq} \equiv 0$ unless $|q| \leq k$.

Another example of a tensor operator is the operator of tensor multiplication by some spin-$k$ multiplet of states $|kq\rangle$, i.e.

$$M(k,q)|\psi\rangle := |kq\rangle|\psi\rangle. \tag{3}$$

**2.** Let $J^2$, $J_z$, and $\Omega$ form a complete commuting set of operators with corresponding eigenstates labeled uniquely by $|\omega j m\rangle$. The matrix elements of the first commutation relation (1) imply that the matrix elements of any irreducible tensor operator $T_{kq}$ have a very special structure in the quantum numbers $m_J$:

$$\langle \omega' j' m'|T_{kq}|\omega j m\rangle = 0 \quad \text{unless} \quad m' = m + q. \tag{4}$$

The matrix elements of the remaining commutation relations (2) imply recursion relations for the matrix elements of $T_{kq}$:

$$a\langle \omega' j' m'|T_{k,q\pm 1}|\omega j m\rangle = b\langle \omega' j', m'\mp 1|T_{kq}|\omega j m\rangle - c\langle \omega' j' m'|T_{kq}|\omega j, m\pm 1\rangle \tag{5}$$

where

$$\begin{aligned} a &= \sqrt{k(k+1)-q(q\pm 1)} && (6) \\ b &= \sqrt{j'(j'+1)-m'(m'\mp 1)} && (7) \\ c &= \sqrt{j(j+1)-m(m\pm 1)} && (8) \end{aligned}$$

**3.** Part 2 implies that the matrix elements $\langle\omega' j' m'|T_{kq}|\omega j m\rangle$ with fixed $\omega' j' \omega j$ are linearly determined recursively by (for example) the nonzero matrix element with maximal $m'$ and $m$. (One need not work out the formula explicitly for each matrix element to see that the elements are so determined.) Thus, the $m'm$ matrix elements of any two irreducible tensor operators are proportional to each other in the sense that

$$\langle\omega_1' j' m'|T^{(1)}_{kq}|\omega_1 j m\rangle = S\,\langle\omega_2' j' m'|T^{(2)}_{kq}|\omega_2 j m\rangle \tag{9}$$

where $S$ is a scalar that depends on $\omega_1', \omega_1, \omega_2', \omega_2, j', j$ and the operator $T_k$ but not $m', m, q$. In writing (9) we have assumed of course that the relevant matrix elements of $T^{(2)}_{kq}$ do not vanish identically.

**4.** The matrix elements of the tensor multiplication operator (3), are just the Clebsch-Gordan coefficients $\langle j'm'|kjqm\rangle$. Choosing $T^{(2)}_{kq} = M_{kq}$ in (9) thus shows in particular that

$$\langle\omega' j' m'|T_{kq}|\omega j m\rangle = \langle\omega' j'||T_k||\omega j\rangle\;\langle j'm'|kjqm\rangle, \tag{10}$$

where $\langle\omega' j'||T||\omega j\rangle$ is called the "reduced matrix element".[1] This is the *Wigner-Eckart theorem.* It states that the matrix elements of an irreducible tensor operator are proportional to the Clebsch-Gordan coefficients, with a factor that depends on $\omega', \omega, j', j$ but not $m', m, q$.

**5.** Although our derivation so far only shows that (10) holds when the Clebsch-Gordon coefficients do not vanish, it actually holds as well when they do. Thus, besides (4), there is a further restriction:

$$\langle\omega' j' m'|T_{kq}|\omega j m\rangle = 0 \quad \text{unless} \quad j' \subset k \otimes j. \tag{11}$$

Equations (4) and (11) are sometimes called *selection rules.*

[1] Sometimes a $1/\sqrt{2j'+1}$ is factored out in the definition of the reduced matrix element.

To see that (10) holds in general one can use the commutation relations (1,2) to show that the set of vectors $\{T_{kq}|jm\rangle\}$ is closed under the action of $J_z$ and $J_\pm$, hence can be decomposed into a set of irreducible representations of the rotation group. In particular,

$$J_z T_{kq}|jm\rangle = (q+m)T_{kq}|jm\rangle, \tag{12}$$

so the decomposition proceeds just as for the product space spanned by the vectors $\{|kq\rangle|jm\rangle\}$. This yields a sum of representations $(k+j)\oplus(k+j-1)\oplus\cdots\oplus|k-j|$. Thus the matrix elements of the tensor operator on the left hand side of (10) do in fact vanish whenever the Clebsh-Gordan coefficients on the right hand side vanish.

**6.** A *vector operator* is a tensor operator with $k = 1$. An example is the position vector $r^a$. The Cartesian components of this vector are $(x, y, z)$. The "spherical components," i.e. those which transform as (2) under rotations, are $r_q = \sqrt{\frac{4\pi}{3}}rY_{1q}(\theta,\phi)$, i.e. $r_0 = r\cos\theta = z$ and $r_{\pm1} = \mp r\sin\theta e^{\pm i\phi}/\sqrt{2} = \mp(x \pm iy)/\sqrt{2}$. More generally, for any vector operator, the relation between spherical and Cartesian components is given by

$$V_0 = V^z, \quad V_\pm = \mp(V^x \pm iV^y)/\sqrt{2}. \tag{13}$$

**7.** The Wigner-Eckart theorem implies, as a special case, that the matrix elements of any vector operator $V^a$ between states of the *same*[2] $j$ are proportional to those of the angular momentum operator $J^a$:

$$\langle\omega' jm'|V^a|\omega jm\rangle = C(\omega' j, \omega j)\ \langle jm'|J^a|jm\rangle. \tag{14}$$

Moreover, the coefficient $C(\omega' j, \omega j)$ is given by

$$C(\omega' j, \omega j) = \langle\omega' jm|\vec{V}\cdot\vec{J}|\omega jm\rangle/j(j+1), \tag{15}$$

for any $m$. To see this, multiply (14) by $\langle\omega jm|J^a|\omega jm''\rangle$ and sum over $m$.) This is called the *projection theorem*. It corresponds to the statement that the components of $\vec{V}$ orthogonal to $\vec{J}$ average to zero.

[2] Note that the restriction to matrix elements between states of the *same* $j$ is in general necessary for (14) to be true, since the matrix elements of $J^a$ between different $j$'s vanish, but those of $V^a$ do not in general.

**8.** A useful application of the projection theorem is to express the magnetic moment of a system in terms of the total angular momentum. Consider for example an atom with many electrons. The magnetic moment is $\vec{\mu} = -(\mu_B/\hbar)\sum_i(\vec{L}_i + g_s\vec{S}_i) =: -(\mu_B/\hbar)[\vec{J} + (g_s - 1)\vec{S}]$, where the sum is over the different electrons, $g_s$ is the electron $g$-factor, and $\vec{J} = \vec{L} + \vec{S}$ is the total orbital plus spin angular momentum of the electrons. The magnetic moment is a vector operator with respect to the total electronic angular momentum $\vec{J}$, so the Wigner-Eckart theorem tells us that, acting with a fixed irreducible representation of this angular momentum, we have

$$\langle \omega' JM' | \vec{\mu} | \omega JM \rangle = -g_J \frac{\mu_B}{\hbar} \langle \omega' JM' | \vec{J} | \omega JM \rangle \tag{16}$$

for some coefficient $g_J$, called the *Landé g-factor*, which depends only on the angular momentum quantum numbers. Indeed, using $\vec{S} \cdot \vec{J} = [J^2 + S^2 - (J - S)^2]/2$, the projection theorem applied to $\vec{\mu}$ yields

$$g_J = 1 + (g_s - 1)\frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}. \tag{17}$$

**9.** Useful fact: the trace $\sum_m \langle \omega jm | T_{k0} | \omega jm \rangle$ of the matrix elements of $T_{k0}$ ($k \neq 0$) in a subspace of given $\omega$ and $j$ is zero. *Proof*: The trace of a commutator of finite dimensional matrices vanishes, and $T_{k0} \propto [J_+, T_{k,-1}]$, which can be truncated to the given subspace since $J_+$ acts within the subspace.

**10.** *Hole-Particle equivalence*: In some ways, a shell filled with identical fermions except for $n$ "holes" behaves the same as a shell with only $n$ such particles. More precisely, let $T_{k0}(i)$ be a single particle irreducible tensor operator with $k > 0$, indexed by the particle label $i$. It can be shown that

$$\langle j^{2j+1-n} JM | \sum_{i=n+1}^{2j+1} T_{k0}(i) | j^{2j+1-n} JM \rangle = (-1)^{k+1} \langle j^n JM | \sum_{i=1}^{n} T_{k0}(i) | j^n JM \rangle, \tag{18}$$

where $|j^n JM\rangle$ is a totally antisymmetric state of $n$ identical fermions, each with angular momentum $j$, adding up to a total angular momentum $J$ and total $z$-component of angular momentum $M$. (For a proof, see for example *Nuclear Shell Theory*, A. de Shalit and I. Talmi (Academic Press, 1963).)